THE MATHEMATIZATION OF THE PHYSICAL SCIENCES - DIFFERENTIAL EQUATIONS OF NATURE

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Summary

From antiquity mathematics has been used to describe and understand nature. During the last centuries before Christ Babylonian mathematicians developed an accurate description of many celestial phenomena. Greek mathematicians developed a different account of the motion of the heavenly bodies and also applied mathematics to optics, statics, and music. These mathematical theories were perfected during the middle ages

and the renaissance. From 1600 many new areas of natural science were mathematized. Galilei and Newton developed the mathematical principles of mechanics, and around 1800 new theories of electrostatics and heat conduction arose. Fluid mechanics and electromagnetism received mathematical treatments later in the 19^{th} century, and the beginning of the 20^{th} century saw the emergence of two revolutionary new mathematically sophisticated physical theories: the theory of relativity and quantum mechanics.

Differential equations have been a particularly efficient tool in the mathematical description of nature and the history of these equations are intimately linked with the development of physics. The first ordinary differential equations and the first techniques for solving them arose at the same time as the differential calculus as a means to solve various problems mostly of a mechanical nature. Later in the 18^{th} century continuum mechanics gave rise to the first partial differential equations. During the 19^{th} century the wave equation, the Laplace equation, the heat equation, the Navier-Stokes equation, and Maxwell's equations were set up and studied as a way to understand various aspects of nature.

At first research on differential equations centered on finding their solutions in finite form, but from the 1820s more qualitative questions such as existence began to be investigated.

1. Everything is Number

From of old humans have attempted to get to grips with their own affairs and with nature surrounding them. This quest has given rise to mathematics. In the earliest written sources from Mesopotamia (Babylon) and Egypt (about 3000 BC.) mathematics was predominantly used for administrative and trade purposes, but also spatial relations and the heavenly phenomena were made the subject of numerical treatment.

The subsequent Greek mathematical culture was from the outset linked to the description of nature. Legend has it that the first known Greek mathematician Thales acquired fame by predicting a solar eclipse in 585 BC and by measuring distances to ships at sea and the height of pyramids. The next known Greek mathematician Pythagoras (ca. 500 BC) developed a thoroughly mathematical philosophy of nature. His motto is said to have been: "Everything is number" by which he meant that everything in the world can be described by natural numbers and ratios between them. He was reportedly convinced about the universal powers of mathematics when he discovered that even esthetical properties such as musical harmonies can be captured by numbers. More precisely he discovered that if a taut string is divided in two, the tones produced by the two sections form a harmonious interval if the lengths of the two sections have a ratio to each other described by small natural numbers. For example if the lengths have a ratio equal to 1:2 the interval is an octave. The ratio 2:3 gives a fifth and 3:4 gives a fourth. Moreover multiplication of the ratios corresponds to addition of the intervals. For example a fifth and a quart give an octave corresponding to: $2/3 \cdot 3/4 = 1/2$.

There is an unbroken tradition from Pythagoras's philosophy of mathematization to present day natural science. It goes via Plato, who according to legend would not let anybody unversed in geometry enter his Academy, over Archimedes, Galilei, and Newton to the highly mathematized theories of modern science. However, many of the mathematical descriptions suggested by the Pythagoreans have later been rejected as mere number mysticism. For example the Pythagoreans argued that there must be 10 heavenly bodies because 10 was a holy number, the sum of the numbers 1,2,3,4, the so-called tetractys.

2. Ancient Astronomy

Soon after the Pythagoreans had formulated their mathematico-mystical world view, two very accurate descriptions of astronomical phenomena were developed, one in Babylon and one in Greece. Based on several thousand years of observations, the late-Babylonian scribes invented numerical procedures for predicting phenomena such as the rising of the moon and the planets, and the occurrence of eclipses. The predictions were based on a set of numerical tables of linear zigzag functions, that we can now interpret as for example the monthly velocity of the sun (or the earth).

Contrary to this arithmetical approach, the Greek astronomical theories were thoroughly geometrical. Aristarchus (c 310-230 BC) reportedly argued for a heliocentric universe but the most accurate antique theories were based on a geocentric model. Eudoxos, Apollonius, and Hipparchus contributed to the construction of the geocentric theory that found its final form in Ptolemy’s Almagest (c 150 AD). According to this theory the planets rotate on a circle (the epicycle **E**) whose centre in turn is carried around on another circle (the deferent **D**) centered at a point **O** which lies half way between the earth **G** and the so-called equant **A** around which the center of the epicycle moves uniformly. Ptolemy adjusted the parameters of this model (such as the radii of the circles and the angular velocities of the circular motions) using ancient Babylonian observations. The resulting description was so accurate that it was used until the renaissance.

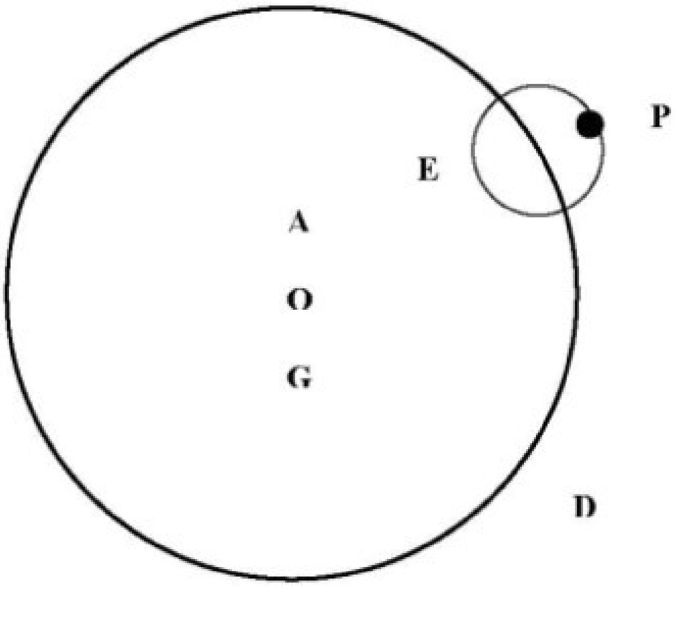

Figure 1. Ptolemaic planetary theory.

In order to deal with astronomical phenomena the Greeks developed trigonometry and spherical geometry. As the basic trigonometric function Hipparchus and Ptolemy chose to tabulate the cord (cord *v*) subtended by a given angle *v* in a circle of radius 60. When trigonometry was transferred to India during the middle ages the cords were replaced by the half chord of the double angle, i.e. the sine. Only with Euler (17th century) was the radius of the circle conveniently reduced to 1.

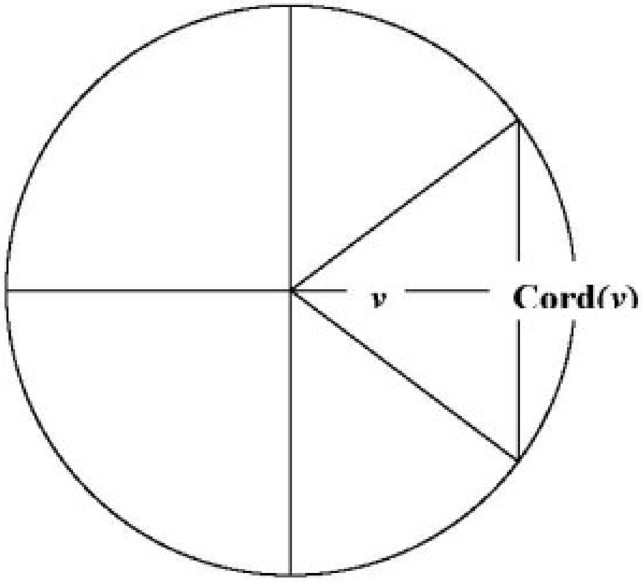

Figure 2. The cord of an angle in a circle of arbitrary radius.

The relation between Greek mathematics and astronomy exemplifies a general pattern: In the process often misleadingly called “application” of mathematics, it is rarely just a question of taking an already developed piece of mathematics and apply it to some other domain for example a natural phenomenon. In most cases the mathematics required for the application did not exist prior to the application, or was insufficiently developed. In these cases new mathematics emerged as a result of the “application”. In this way both mathematics and the area of application, such as the natural sciences, typically benefit from such an encounter. The rest of this article will show numerous examples of this cross fertilization.

3. Optics and Statics

In addition to astronomy (and music) two other natural sciences were mathematized in Greek antiquity: Optics and statics. The first mathematical text on optics was written by Euclid (c 300 BC), the author of the more famous *Elements* in which he collected much of the “elementary” mathematical knowledge of his time. Following the model of the Elements, Euclid tried to give an axiomatic treatment of vision assuming for example that the apparent size of an object is determined by the angle subtended by the visual cone from the eye. About a century later Diocles used the law of reflection (the equality of the angle of incidence and the angle of reflection) to establish that a parabolic mirror would reflect rays parallel to its axis to one point, its focus. Hero (c 10-70 AD) deduced the law of reflection from the assumption that light follows the shortest paths and Ptolemy studied the refraction of light on the interface between two media (air and water). However the law of refraction eluded him. It was discovered by the Islamic mathematician Ibn Sahl and independently by three early 17th century mathematicians: Harriot, Descartes and Snell, after whom it is often called.

Statics, or the theory of equilibrium of bodies at rest, was made the subject of two very influential works by Archimedes (c 287- 212 BC). In the paper *On the Equilibrium of Planes* (in two parts or books) he set up a system of axioms from which he deduced the law of the lever and determined the centers of gravity of a triangle, a trapezium and a parabolic segment. In the paper *On Floating Bodies* (also in two parts) he proved that a liquid earth must be spherical, he derived Archimedes's law, and he showed how a solid body shaped as e.g. the section of a circle or the section of a parabola will orient itself if it floats on water.

4. The Middle Ages and the Renaissance

During the Middle ages Arabic mathematicians refined the Greek applications of mathematics both to optics (e.g. Ibn al-Haytham (965-1040)) and to astronomy, and in the renaissance Nicolaus Copernicus (1473-1543) revolutionized cosmology by replacing Ptolemy's geocentric universe with a heliocentric one. However, Copernicus continued the Ptolemaic tradition of describing the planetary orbits by circular motions. Thus from a mathematical point of view Johannes Kepler (1571-1630) was more innovative when he, on the basis of Tyco Brahe's observations, suggested that planets describe ellipses with the sun in one focus and move such that the areas swept out by the radius vector from the sun to the planet are the same in equal times (1609). Kepler also broke old barriers when he began to speculate about the physical causes of the observed motions. All his predecessors had treated astronomy as a mathematical science that had noting to do with the physical theory of the motion of bodies near the earth.

5. Mechanics of Motion

Motion of earthly bodies had been discussed by the Greek philosopher Aristotle (384-322 BC) and his theory received a mathematical treatment during the late medieval period in the so-called Merton school. But the great break through in the theory of motion (kinematics) came with Galileo Galilei (1564-1642). In his *Discussion and mathematical demonstrations concerning two new sciences* (1638) he argued that free fall is a constantly accelerated motion and he derived some of its properties. For example he showed that when a body falls from rest the distances traveled during two time intervals are to each other as the squares of those times. From this he also deduced that a body (for example a cannonball) that has a uniform horizontal motion and a constantly accelerated downward motion will follow a parabolic path. More important than any particular result was Galilei's insistence that the book of nature is written in the language of mathematics. His program for studying nature was a continuation of Archimedes's mathematical theory of statics, and struck a happy medium between Baconian empiricism and Cartesian rationalism.

6. Newtonian Mechanics

If Galilei was the father of kinematics, Isaac Newton (1642-1727) was the father of dynamics, i.e. the study of motion and the forces that cause it. Combining Galilei's theory of motion of earthly bodies with Kepler's laws for the motion of the heavenly bodies he created a unified mathematical approach to natural phenomena that remained the paradigm for the following centuries. He published his ideas in the *Principia Mathematica Philosophia Naturalis* (The mathematical Principles of Natural Philosophy) (1687) starting with his three celebrated laws of motion, the second of which stated that the change of motion is proportional to the impressed force (later reformulated as: force is equal to mass times acceleration). From these laws and Kepler's laws he deduced that the force keeping a planet in its orbit was a central attractive force between the sun and the planet varying inversely as the square of the distance between them. He further argued that this force was a universal force (gravitation) that acted between all bodies, be they planets, the sun, the earth, the moon or an apple.

Having established this universal law of gravitation Newton could then show that it gave an accurate account of all the heavenly motions, not only the approximate motions expressed by Kepler's laws but also the small irregularities that Newton could explain as resulting from the mutual interactions between the planets.

Newton presented his synthesis of the system of the world in a geometric language. But in his own analyses and his semi-public manuscripts he also made use of René Descartes' analytical geometry (published 1637), in which geometric problems are studied using algebraic techniques, and of the fluxional calculus that he had developed in 1666. This latter theory dealt with variable quantities and their rate of change or velocities. If x and y denote quantities that vary with time (e.g. the distances between some heavenly bodies) then Newton denoted their rate of change or their velocities by $\dot{x}$ and $\dot{y}$. He called these the fluxions of the fluents x and y. He formulated two main problems for the study of fluxions: 1. Given an equation involving two fluents, find the ratio between their fluxions, and conversely: 2. Given an equation involving two fluents and their fluxions, find the equation between the fluents. The former problem corresponds to differentiation and the latter to what we would today call the solution of a differential equation. Newton used the fluxional calculus to solve many problems related to curves considered as trajectories of a moving point.

7. Early Differential Equations

Ten years after Newton had developed the fluxional calculus the German diplomat, philosopher and mathematician Gottfried Wilhelm Leibniz (1646-1716) independently invented his differential calculus. If x and y are two variable quantities for example the x and y coordinates of a point on a plane curve he let dx and dy denote the infinitely small increments or differentials by which they vary. The differential calculus consisted of rules for calculating with such differentials. It was quickly taken up by the brothers Jacob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748) and other continental mathematicians. Just as Newton's fluxional calculus, the differential calculus was from the beginning used to solve problems leading to differential equations.

In the language of the 17th and 18th centuries a differential equation is an equation involving differentials e.g. dx and dy (and possibly higher order differentials), and solving (or integrating) it means to determine an equation between the variables themselves. In the later terminology of Lagrange and Cauchy, that we still use to day, a differential equation is an equation involving an unknown function f and some of its derivatives $f', f'', \ldots$ and solving (or integrating) it means to determine a function (or all the functions) that substituted for f will make the equation hold true. The order of a differential equation is the maximal number of times the unknown function is differentiated.

Most of the early examples of differential equations had their origin in mechanical problems. The earliest example was the isochrone problem that asks for a curve along which a body descending under the influence of gravity will reach the bottom point in the same amount of time from whichever point on the curve the descent begins. Christiaan Huygens (1624-1695) had already in 1659 shown that the curve is a cycloid, i.e. the path of a point on a wheel rolling on a straight line without slipping. He had used this insight in his construction of the first pendulum clock. In 1690 Jacob Bernoulli found the same result by setting up a differential equation for the isochrone and solving it. The following year and using similar techniques, his brother and other mathematicians determined the catenary, i.e. the curve assumed by a flexible inelastic hanging cord or chain. He showed that the curve satisfies the differential equation $dy/dx = s/a$ where s denotes the curve-length along the curve. Through algebraic manipulations of this equation and one integration he transformed it into the differential equation

$$dx = \frac{ady}{\sqrt{y^2 - a^2}},$$

which he solved by a geometric construction. Today we recognize the solution as the graph of the hyperbolic cosine function.

8. The Brachistochrone

Other mechanical problems solved using differential equations during the late 17th and early 18th centuries include the shape of a square sail under the influence of the wind and the shape of such a sail filled with water. But the most far reaching of these problems was the brachistochrone problem posed by Johann Bernoulli in 1696 and solved the following year by himself, his brother, Newton and Leibniz. Given two points in a plane; the problem asks for the curve between them such that a body sliding along the curve under the influence of gravity will travel in the shortest time from the higher point to the lower one.

Johan Bernoulli's solution combined optics and mechanics. Indeed, Pierre de Fermat (1601-1665) had deduced Snell's law of refraction from the assumption that light travels along the quickest path and has lower speed in denser materials. Moreover Galilei had shown that a body falling from rest will have a speed proportional to $\sqrt{x}$ where x is the vertical distance the body has fallen, irrespective of the path along which it has reached the position. Thus Bernoulli could consider the sliding body as a light particle moving in a horizontally layered medium with the velocity of light increasing downward as $\sqrt{x}$.

The light ray would then follow the brachistochrone. Expressing the sines in Snell's law in terms of the differentials dx and dy led him to the differential equation

$$dy = dx\sqrt{\frac{x}{a-x}}.$$

This equation has its variables separated and can therefore be solved simply by taking integrals on both sides. However, Bernoulli immediately recognized it as the differential equation of the cycloid and had thereby established that the brachistochrone is also a cycloid.

Jacob Bernoulli's solution was less elegant but went deeper. He expressed the time used by the body to slide along a given curve $y(x)$ as an integral involving $y(x)$ and its differential, and succeeded in determining the curve that minimizes this integral. This can be seen as the beginning of the calculus of variations in which the aim is to determine that function among a given set of admissible functions which minimizes a given integral involving the function and its derivatives (or more generally a functional, i.e. a function of functions). Variational problems later became very important in mechanics and other areas of physics and other sciences.

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Biographical Sketch

Jesper Lützen (born 1951 in Denmark) received his education in mathematics, physics and the history of mathematics at the University of Aarhus and later taught at University of Odense. He is currently professor of the history of mathematics at the Department of Mathematical Sciences of the University of Copenhagen and a member of the Royal Danish Academy of Sciences and Letters. His research mostly deals with the history of 19th century mathematics and often highlights the connection between mathematics and physics. His main works are the three books: *The Prehistory of the Theory of Distributions*, Springer Verlag, New York 1982, *Joseph Liouville 1809--1882: Master of Pure and Applied Mathematics*, Springer-Verlag, New York 1990, and *Mechanistic Images in Geometric Form: Heinrich Hertz's Principles of Mechanics*, Oxford University Press, 2005. He is currently engaged in studying the history of impossibility theorems in mathematics, in particular the impossibility of the classical construction problems.