MODEL REDUCTION

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Keywords: Model reduction, Model approximation, Linear systems

Contents

Summary

Modeling physical systems usually results in complex high-order dynamic models. It is

often desirable to replace (approximate) these models by simpler models with reduced order. In this process it is important to *design* the reduced model so as to capture the *important* properties of the original high-order model. This chapter describes some procedures that are available for the model reduction of linear time-invariant systems.

1. What is Model Reduction?

The description of a dynamic physical system by a set of differential (or difference) equation is a very useful tool in science. These equations, refereed here as a mathematical *model*, can be obtained from basic physical principles or as a result of experiments. A measure of the "complexity" of the system model is the number of first order equations used to describe it. This number is often referred as the *order* of the model. Models with elevated order are able to describe very complex phenomena. Consequently, models with high order may be required in order to provide an accurate description of a dynamic system. For instance, models with an infinite number of differential equations often appear in several fields. To name one, the behaviour of materials based on continuum physics is often described by partial differential equations or by an infinite number or ordinary differential equations.

If the capacity of a model to accurately describe a system seems to increase with the order of the model, in practice, models with low orders are required in many situations. In some cases, the amount of information contained in a complex model may obfuscate simple, insightful behaviors, which can be better captured and explored by a model with low order. In cases such as control design and filtering, where the design procedures might be computationally very demanding, limited computational resources certainly benefit from low order models. These examples justify the need to develop procedures that are able to approximate complex high order models by generating adequate *reduced order models*. As a result, some degree of detailing will be permanently lost in the reduced order model. The differences between the dynamics of the high order model and the obtained low order model (the *unmodeled* dynamics) can be often taken into account in the low order model as a *noise*, which can be handled using stochastic process methods. In any case, the model reduction procedures might be flexible enough to let the user indicate the essential behaviors that need to be captured for its application.

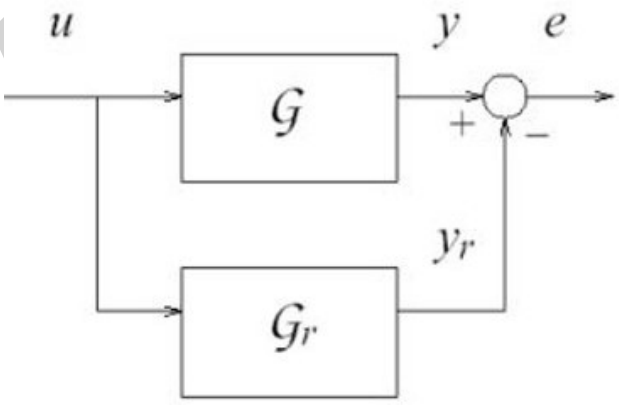

Figure 1: Single Component Model Reduction

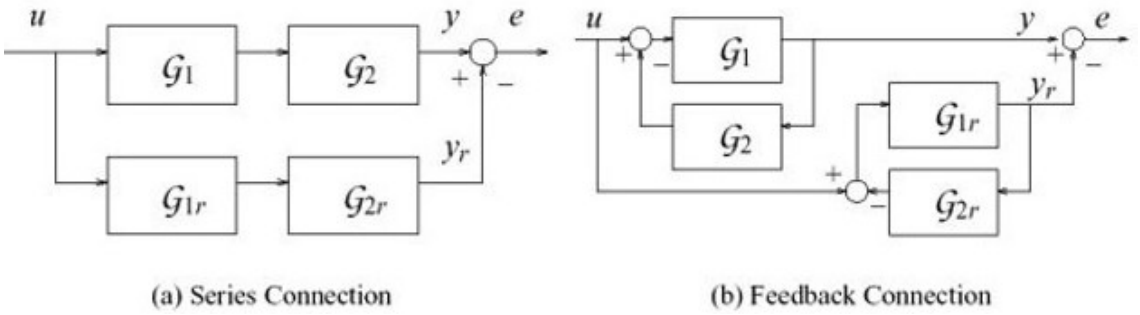

Figure 2: Multi-Component Model Reduction

1.1. Single Component Model Reduction

Given a dynamic system model $\mathcal{G}$ of usually high order *n*, a model reduction method is a procedure that yields some approximated model $\mathcal{G}_r$ of order $n_r < n$. The quality of the approximation is usually evaluated by looking at the *model reduction error*, that is, the signal obtained as the difference between the outputs of the original system and the outputs of the reduced order model driven by the same input signal. That arrangement is depicted in Figure 1. With respect to this figure, the single component model reduction problem can then be loosely stated as follows:

Given a system $\mathcal{G}$, choose a procedure that yields a reduced order model $\mathcal{G}_r$ so that the model reduction error is small.

1.2. Multi-Component Model Reduction

Quite often model reduction procedures might be applied to one component of a system while the other component to which it is connected remain the same. Two examples of such occasion are given. Figure 2 (a) characterizes two components of a system interconnected in series. As an illustration, if this system represents a spacecraft, component $\mathcal{G}_1$ might represent a solar panel while component $\mathcal{G}_2$ could be the body of the spacecraft. The solar array and the body of the spacecraft might be manufactured by different companies and it would be useful to know how to write contracts so that the two companies can manufacture products with *component* dynamic properties such that the connections of the two components produce *system* dynamic properties that meet stringent specifications. The contracts must also characterize through the reduced order models $\mathcal{G}_{1r}$ and $\mathcal{G}_{2r}$ the accuracy required of the models of each component when measured from the system of connected components. Notice that *weighted* single-component model reduction problems are defined by fixing component $\mathcal{G}_1$ (for input weighting) or component $\mathcal{G}_2$ (for output weighting). Several weighted model reduction procedures are available in the literature.

Of course, one can have more complex arrangements of components than the simple example of Figure 2(a). For instance, in Figure 2 (b), a second example of multi-component model reduction involves feedback. In a typical controlled system,

component $\mathcal{G}_1$ is usually the plant and component $\mathcal{G}_2$ is the controller. Both plant and controller might be subject to order reduction, where the reduced order models are represented in the picture by $\mathcal{G}_{1r}$ and $\mathcal{G}_{2r}$.

The general multi-component model reduction problem that includes all the special cases mentioned above can be loosely stated as follows

Given N system components $\mathcal{G}_i, i = 1,..., N$ and an interconnection architecture (as in Figures 1-2 or other), chose procedures for each component that yield reduced order models $\mathcal{G}_{ir}, i = 1,..., N$ so that the overall system error (characterized by e in Figure 1-2) is small.

It is important to note that model reduction methods that make the single-component model reduction error small for some component in the system do not necessarily yield small errors in an interconnected architecture. Conversely, a given reduced order model of a single component might produce unbounded single-component error and small multi-component error. In other words, the architecture of components tends to have a major impact on errors and on the selection of adequate model reduction procedures. One consequence of this fact in the context of control synthesis is that the determination of the model of the plant (in this case also affected by the model reduction procedure) and the design of the control law *are not independent problems*. The recognition of this fact (see the references for examples) has led to a large research effort devoted to the integration of the disciplines of model identification and control design.

1.3. The Quality of the Reduced Order Model

Whenever the system $\mathcal{G}$ and $\mathcal{G}_r$ can be interpreted as operators, the norm of the difference between $\mathcal{G}$ and the reduced order model $\mathcal{G}_r$ may be useful measure of the size of the model reduction error. In the statements of both single- and multi-component model reduction problems, the statement that the model reduction error should be kept small can be quantified through the scalar $\left\| \mathcal{G} - \mathcal{G}_r \right\|$. If such objective is accomplished, it is expected that the error signal e resulting from the connections depicted in Figures 1 and 2 be small for input signals u in some well defined class.

Quantities other than norms can also be used in model reduction. Indeed, given a model for a physical system, it is usually possible (and sometimes very useful) to characterize it in terms of its response to certain input signals. For instance, a linear system model can be completely characterized by its impulsive response, and a number of moments (derivatives) of the impulse response evaluated at a given instant might capture important features of the original model. Hence, *matching* certain properties constitute an alternative model reduction criterion than keeping norms small. In fact, given a physical plant, it is not even necessary to have a complete model in hand to be able to perform model reduction (or identification), since the response of the system to properly generated inputs might be evaluated experimentally. Therefore, a reduced order model can be designed (identified) to match certain frequency or time response properties.

1.4. Characterization of the Single-Component Model Reduction Error

Most parts of this chapter will focus on the single-component model reduction problem. This problem is the most elementary model reduction problem and yet displays the essential mathematical concepts encountered in the more complex multi-component model reduction problem.

It will be assumed that $\mathcal{G}$ is a linear, continuous-time and time-invariant model of order $n < \infty$ described by the set of equations

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = 0 \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

where the $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^q$ is the system output. For simplicity, the dependence of these vectors with respect to the independent variable *t* will be omitted whenever possible. References to the state-space realization (1-2) will frequently appear denoted by the quadruple of matrices (*A*, *B*, *C*, *D*).

The reduced order model $\mathcal{G}_r$ to be determined has the same structure as $\mathcal{G}$, that is, it is a linear, continuous-time and time-invariant model described by

$$\dot{x}_r = A_r x_r + B_r u, x_r(0) = 0 \tag{3}$$

$$y_r = C_r x_r + D_r u, \tag{4}$$

where the $x_r \in \mathbb{R}^{n_r}$ is the reduced order state and $y_r \in \mathbb{R}^q$ is the output of the reduced order model.

In order to emphasize the fact that these systems are linear, they will be henceforth denoted by $\mathcal{G}(s)$ and $\mathcal{G}_r(s)$, respectively. Where the complex variable s alludes to the possibility of computing a Laplace transform (frequency-domain) representation of systems (1-2) and (3-4).

The connection of $\mathcal{G}(s)$ and $\mathcal{G}_r(s)$ as in Figure 1 produces the model reduction error signal $e := y - y_r$. The relation between the common input signal u and e can be described by defining the augmented state

$$\tilde{x} := \begin{pmatrix} x \\ x_r \end{pmatrix}, \tag{5}$$

so that the connection of the system $\mathcal{G}(s)$ and $\mathcal{G}_r(s)$ as in Figure 1 produces the linear time-invariant system

$$\dot{\tilde{x}} = \mathcal{A}\tilde{x} + \mathcal{B}u, \tilde{x}(0) = 0 \tag{6}$$

$$e = \mathcal{C}\tilde{x} + \mathcal{D}u \tag{7}$$

defined with the following matrices

$$\mathcal{A} := \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B \\ B_r \end{bmatrix}, \quad \mathcal{C} := [C \ -C_r], \quad \mathcal{D} := D - D_r \tag{8}$$

The transfer function from the input u to the model reduction error signal e in the *error system* (6-8) is denoted by $\mathcal{E}(s)$. Notice that $\mathcal{E}(s) = \mathcal{G}(s) - \mathcal{G}_r(s)$.

2. Linear System Properties

This section introduces several concepts and properties associated with linear systems that are of interest of the model reduction problem. It intends to summarize some important results that will be used in the model reduction methods to be described in Sections 3 and 4. (See Description and Classification, System Characteristics)

2.1. Input-Output Transfer Function

Given an arbitrary input signal $u(t)$, the value of the output signal $y(t)$ of the linear system (1-2) can be calculated by the convolution integral

$$y(t) = \int_0^\infty g(t-\tau)u(\tau)d\tau \tag{9}$$

where $g(t)$ is a function that describes the response of system (1-2) to independent impulsive inputs at all input channels, that is, the linear system *impulse response*. That relation can be equivalently characterized in somewhat simpler form

$$\mathcal{Y}(s) = \mathcal{G}(s)\ \mathcal{U}(s), \tag{10}$$

where $\mathcal{Y}(s), \mathcal{U}(s)$ and $\mathcal{G}(s)$ denote the *Laplace transform* of, respectively, the output $y(t)$, the input $u(t)$ and the impulse response $g(t)$. In particular, it can be shown that, for the linear system (1-2), the impulse response $g(t)$ and its associated *transfer function* $\mathcal{G}(s)$ are given by

$$g(t) := Ce^{At}B + D\delta(t), \quad \mathcal{G}(s) := C(sI - A)^{-1}B + D. \tag{11}$$

The transfer function $\mathcal{G}(s)$ is a rational function of the complex variable s and provides a frequency-domain description of the input-output behavior of the system (1-2).

From (11), more than one realization (A, B, C, D) of a linear system can produce the same impulse response $g(t)$ and transfer function $\mathcal{G}(s)$. That is, different system

realizations can produce the same input-output behavior. In particular, all linear systems whose coordinates are related by

$$x = Tz, \quad z = T^{-1}x. \tag{12}$$

where the square matrix T is nonsingular, share the same input-output transfer function. Such systems are said to be related by a *similarity transformation* which is completely characterized by matrix T. Notice that the transformed system produced by (12) has the state-space representation

$$\dot{z} = T^{-1}ATz + T^{-1}Bu,\ z(0) = 0 \tag{13}$$

$$y = CTz + Du, \tag{14}$$

and that

$$\mathcal{G}_T(s) = (CT)(sI - T^{-1}AT)^{-1}(T^{-1}B) + D = C(sI - A)^{-1}B + D = \mathcal{G}(s) \tag{15}$$

which is indeed independent of the choice of similarity transformation matrix T.

2.2 Controllability and Observability

The following concepts play an important role in the analysis of linear systems.

Definition 1 Given the pair of matrices (A, B) where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ the following statements are equivalent:

a) (A, B) is *controllable*,

b) There exists no scalar $\lambda \in \mathbb{C}$ and no vector $v \in \mathbb{C}^n \neq 0$ such that

$$v^*(\lambda I - A) = 0, \quad v^*B = 0, \tag{16}$$

c) The *controllability matrix*

$$W_c := \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} \tag{17}$$

has rank n.

Definition 2 Given the pair of matrices (A, C) where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{r\times n}$ the following statements are equivalent:

a) (A,C) is *observable*,

b) There exists no scalar $\lambda \in \mathbb{C}$ and no vector $v \in \mathbb{C}^n \neq 0$ such that

$$(\lambda I - A)v = 0, \quad Cv = 0, \tag{18}$$

c) The *observability matrix*

$$W_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{19}$$

has rank *n*.

By extension, a realization (*A*, *B*, *C*, *D*) is said to be controllable if the pair (*A*, *B*) is controllable and observable if the pair (*A*, *C*) is observable.

For asymptotically stable systems, controllability and observability can be equivalently checked by solving the Lyapunov equations

$$AP_c + P_c A^T + BB^T = 0, \tag{20}$$

$$A^T P_o + P_o A + C^T C = 0. \tag{21}$$

The solutions P_c and P_o are called, respectively, controllability and observability Grammians. The following lemmas are standard.

Lemma 1 *The controllability Grammian* P_c *is positive definite if, and only if, matrix A is asymptotically stable and* (*A, B*) *is controllable.*

Lemma 2 *The observability Grammian* P_o *is positive definite if, and only if, matrix A is asymptotically stable and* (*A, C*) *is observable.*

It is worth noticing that the Grammians are not realization independent since

$$P_{cT} = T^{-1} P_c T^{-T}, \quad P_{oT} := T^T P_o T. \tag{22}$$

However, the product of the Grammians, that is,

$$P_{cT} P_{oT} = T^{-1} P_c P_o T \tag{23}$$

possesses invariant eigenvalues.

2.3. Frequency Moments and Markov Parameters

Assume that the transfer function $\mathcal{G}(s)$ is strictly proper (D=0) and analytic on the

imaginary axis. Its Taylor power series expansion around $S = j\omega$ provides

$$\mathcal{G}(s) = C(sI - A^{-1})B = \sum_{i=0}^{\infty} M_i(j\omega)\ (S - j\omega)^i\ , \tag{24}$$

where

$$M_i(j\omega) := C(j\omega I - A)^{-(i+1)} B,\ \ i = 0,1,... \tag{25}$$

The matrices $M_i(j\omega), 0 \le \omega < \infty$, are known as *low frequency moments* of the transfer function $\mathcal{G}(s)$. The *high frequency* moments

$$M_i(j\infty) := \lim_{\omega\to\infty} M_i(j\omega) = CA^i B,\ \ i = 0,1,..., \tag{26}$$

are known as *Markov parameters*. In single input systems, the Markov parameters can be given a physical interpretation by applying an unitary impulse at the input channel. Using (9-11), such input produces the output

$$y(t) = \int_0^{\infty} Ce^{A(t-\tau)} B\delta(\tau) d\tau = Ce^{At} B \tag{27}$$

Therefore, Markov parameters are associated with the *i*th derivative (*time moment*) of the impulse response at instant zero

$$\left.\frac{d^i y(t)}{dt^i}\right|_{t=0} = M_i(j\infty), i = 0,1,... \tag{28}$$

Notice that the frequency moments are input-output properties and should remain invariant under a similarity transformation. Indeed, the low frequency moments (25) are such that

$$\begin{aligned} M_i(j\omega)_T &= (CT)(j\omega I - T^{-1}AT)^{-(i+1)}(T - B) = \\ &C(j\omega I - A)^{-(i-1)} B = M_i(j\omega) \end{aligned} \tag{29}$$

for all $i = 0,1,...$ The same pattern can be used to shown that the Markov parameters are also invariant.

2.4. Output Correlation and Power Moments

Another quantity related to the input-output behavior of a linear system is the deterministic *output correlation* for impulsive inputs (white noise inputs in the stochastic case). Assume that the linear model (1-2) is asymptotically stable and strictly proper (D=0). The output correlation for impulsive inputs is defined by

$$R(t) = \sum_{i=1}^{m} \int_0^\infty y^i(t+\tau) y^{i^T}(\tau) d\tau \,, \tag{30}$$

where $y^i(t),\ i = 1,...,m$ denotes the output of the system due to an impulse applied at the *i*th input channel. It can be shown that (30) can be computed as

$$R(t) = Ce^{At} P_c C^T \,, \tag{31}$$

where P_c is the controllability Grammian, i.e., the positive semidefinite solution of the Lyapunov equation (20). Following Section 2.3, the output covariance (31) can be Laplace transformed and expanded in Fourier series

$$\mathcal{R}(s) = C(sI - A)^{-1} P_c C^T = \sum_{i=0}^{\infty} R_i(j\omega)\,(s - j\omega)^i \,. \tag{32}$$

The matrices

$$R_i(j\omega) = C(j\omega I - A)^{-(i+1)} P_c C^T, \quad i = 0,1,... \tag{33}$$

are known as the *low frequency power moments*. The *high frequency* moments

$$R_i(j\infty) := \lim_{\omega \to \infty} R_i(j\omega) = CA^i P_c C^T, i = 0,1,..., \tag{34}$$

are called *covariance parameters*.

The same reasoning used to show that the frequency moments and Markov parameters are independent of state-space realizations can be used to show that the power moments $R_i(j\omega),\ i = 0,1,...,$ are also invariant under a similarity transformation.

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Biographical Sketches

Robert E. Skelton began his career at the Marshall Space Flight Center, working first with Lockheed Missiles and Space Company and then Sperry Rand for 12 years. From 1975–1996, he was a professor of aeronautics and astronautics at Purdue University. He is a Fellow of AIAA and IEEE. He was the 1991 Russell Severence Springer Professor, UC Berkeley. For five years, he served on the National Research Council's Aeronautics and Engineering Board. He served on the External Independent Review Team for the second servicing mission of the Hubble Space Telescope, and is now serving on this team for the next servicing mission. He has published three books and over a hundred journal papers. He currently heads the Structural Systems and Control Lab at University of California, San Diego, and researches the integration of control and plant design. This includes integrating signal processing and control, integrating modeling and control, integrating structure design and control. His interest in controlled tensegrity structures is motivated by biological systems.

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