EIGENSTRUCTURE ASSIGNMENT FOR CONTROL

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Contents

Summary

Eigenstructure assignment is considered for the controller design of MIMO systems. The definition of eigenstructure assignment, the role of the system eigenstructure, the freedom for eigenstructure assignment, the allowable eigenvector subspaces, and the calculation of feedback controllers are discussed. Several eigenstructure assignment techniques are detailed. for example, assignment of desired eigenvectors, comprise between eigenvalues

and eigenvectors, and multiobjective eigenstructure assignment. A brief introduction to various eigenstructure assignment methods is given: *e.g.* basic eigenstructure assignment, recursive eigenstructure assignment, low sensitive eigenstructure assignment, robust eigenstructure assignment, and eigenstructure assignment for descriptor systems and dynamical compensator systems.

1 Introduction

In the 1960s, Wonham presented the fundamental result on eigenvalue assignment in linear time-invariant multivariable controllable systems. This states that the closed-loop eigenvalues of any controllable system may be arbitrarily assigned by state feedback control. Later, Moore found that degrees of freedom are available over and above eigenvalue assignment using state feedback control for linear time-invariant multi-input multi-output (MIMO) systems. Since then, numerous methods and algorithms involving both state and output feedback control have been developed to exercise those degrees of freedom to give the systems some good performance characteristics. Some 20 years ago, eigenstructure assignment was put aside in favor of frequency domain methods as, at that time, it was a rather limited approach to multivariable control system design. Eigenstructure assignment was then rather limited to state feedback, and output feedback design methods were just beginning. Little attention had been paid to sensitivity minimization, control system robustness and dynamical compensator design. The research field of eigenstructure assignment now covers all of these topics and the subject is certainly mature enough for real application. It is also relatively easy to understand and use; control laws designed via eigenstructure assignment are fully implementable and do not have high order realizations.

2. Definition of Eigenstructure Assignment

Consider a linear MIMO time-invariant state space control system

$$\delta x = Ax + Bu \qquad (1)$$

$$y = Cx \qquad (2)$$

where δx represents $\dot{x}(t)$ for continuous systems and $x(t+1)$ for discrete systems, $x \in \mathcal{R}^n$ the state vector, $u \in \mathcal{R}^m$ the control input vector, $y \in \mathcal{R}^s$ the output vector, and, $A \in \mathcal{R}^{n\times n}$ $B \in \mathcal{R}^{n\times m}$ and $C \in \mathcal{R}^{s\times n}$ the system matrices.

Without loss of generality, the following assumption is made for the system Eqs. (1) and (2): rank(B) = m, rank(C) = s, and uncontrollable and/or unobservable eigenvalues (or poles) are stable. If either of the first two conditions is violated, some transformations are needed to obtain an equivalent system, which satisfies the conditions.

The linear output feedback control law is applied to the open-loop system given by Eqs. (1) and (2):

$$u = Ky \tag{3}$$

where $K \in \mathcal{R}^{m \times s}$. This results in the closed-loop system

$$\delta x = \left(A + BKC \right) x \tag{4}$$

Now, let us define a closed-loop self-conjugate eigenvalue set

$$\Lambda = \left\{ \lambda_i : \lambda_i \in \mathcal{C}, i = 1, 2, ..., \tilde{n} \right\} \tag{5}$$

i.e., the set of the eigenvalues of the closed-loop matrix $A + BKC$, where $\tilde{n}$ is the number of distinct eigenvalues, and $\mathcal{C}$ denotes the complex space. For stability of the closed-loop system, λ_i must be in the open complex left plane for continuous systems and in the open unit circle for discrete systems. For an uncontrollable and/or unobservable system, the uncontrollable and/or unobservable open-loop eigenvalues should be included in the closed-loop self-conjugate eigenvalue set Λ. Though state- and output-feedback eigenstructure assignment cannot change those eigenvalues, their corresponding eigenvectors may properly be chosen to improve the insensitivity of the closed-loop matrix and the robustness of the closed-loop system.

For single closed-loop eigenvalues, the corresponding right and left eigenvectors r_i and l_i, respectively, of the i -th eigenvalue λ_i are defined as

$$\left(\lambda_i I - A - BKC \right) r_i = 0 \tag{6}$$

$$l_i^T \left(\lambda_i I - A - BKC \right) = 0 \tag{7}$$

where $r_i, l_i \in \mathcal{C}^{n \times 1}$.

For multiple closed-loop eigenvalues, denote the algebraic and geometric multiplicity of the i -th eigenvalue λ_i by q_i and s_i, respectively. Then, in the Jordan canonical form of the matrix $A + BKC$, there are s_i Jordan blocks, associated with the i -th eigenvalue λ_i, of orders p_{ij}, $j = 1, 2, ..., s_i$, and the following relations:

$$\sum_{j=1}^{s_i} p_{ij} = q_i \tag{8}$$

$$\sum_{i=1}^{\tilde{n}} q_i = n \tag{9}$$

Now, let the right eigenvectors and generalized eigenvectors of the matrix $A + BKC$ corresponding to the eigenvalue λ_i be $r_{ij,k} \in \mathcal{C}^{n \times 1}$, $k = 1, 2, ..., p_{ij}$, $j = 1, 2, ..., s_i$. The right eigenvector and generalized eigenvector for a multiple eigenvalue are defined as

$$\left(\lambda_i I - A - BKC\right) r_{ij,k} = -r_{ij,k-1}, \quad r_{ij,0} = 0 \tag{10}$$

for k = 1, 2, ..., p_{ij}, j = 1, 2, ..., s_i and $i = 1,2,...,\tilde{n}$. Thus, the right generalized eigenvector matrix is given by

$$R = \left[R_1, R_2, ..., R_{\tilde{n}} \right] \in \mathcal{C}^{n\times n} \tag{11}$$

$$R_i = \left[R_{i1}, R_{i2}, ..., R_{is_i} \right] \in \mathcal{C}^{n\times q_i} \tag{12}$$

$$R_{ij} = \left[r_{ij,1}, r_{ij,2}, ..., r_{ij,p_{ij}} \right] \in \mathcal{C}^{n\times p_{ij}} \tag{13}$$

for j = 1, 2, ..., s_i, $i = 1,2,...,\tilde{n}$ where, in fact, R_i contains all the right eigenvectors and generalized eigenvectors associated with the eigenvalue λ_i. Similar to the definition of the right eigenvectors and the generalized eigenvectors, the left eigenvectors and the generalized eigenvectors $l_{ij,k} \in \mathcal{C}^{n\times 1}$ for multiple eigenvalues are defined by

$$l_{ij,k}^T \left(\lambda_i I - A - BKC\right) = -l_{ij,k-1}^T, \quad l_{ij,0} = 0 \tag{14}$$

for k = 1, 2, ..., p_{ij}, j = 1, 2, ..., s_i and $i = 1,2,...,\tilde{n}$. Then, the left generalized eigenvector matrix is given by

$$L = \left[L_1, L_2, ..., L_{\tilde{n}} \right] \in \mathcal{C}^{n\times n} \tag{15}$$

$$L_i = \left[L_{i1}, L_{i2}, ..., L_{is_i} \right] \in \mathcal{C}^{n\times q_i} \tag{16}$$

$$L_{ij} = \left[l_{ij,1}, l_{ij,2}, ..., l_{ij,p_{ij}} \right] \in \mathcal{C}^{n\times p_{ij}} \tag{17}$$

for j = 1, 2, ..., s_i, $i = 1,2,...,\tilde{n}$, where, the matrix L_i consists of all left eigenvectors and generalized eigenvectors associated with the eigenvalue λ_i.

Eigenstructure assignment may therefore be described simply as the assignment of the eigenvalues and eigenvectors (including right eigenvectors, left eigenvectors and generalized eigenvectors) of the closed-loop matrix using the linear control law given by Eq. (3). It consists, essentially, of the following steps:

- Choose a set (or sets) of possible closed-loop eigenvalues (or poles).
- Compute the associated so-called allowable eigenvector subspaces, which describe the freedom available for closed-loop eigenvector assignment.
- Select specific eigenvectors from the allowable eigenvector subspaces according to some design strategies.

- Calculate a control law, appropriate to the chosen eigenstructure.

Eigenstructure assignment is a design technique which may be used to assign the entire eigenstructure (eigenvalues, and right or left eigenvectors) of a closed-loop linear system via a feedback control law.

3. Role of the System Eigenstructure

The definition of eigenstructure assignment has been given, and now the significance of the eigenstructure in terms of the system time response is outlined.

For the sake of simplicity, it is assumed that all eigenvalues are real and distinct. Denote the i -th eigenvalue and corresponding right and left eigenvectors of the system described by λ_i, r_i and l_i, respectively. The spectral and modal matrices can then be defined as

$$R = [r_1,\ r_2,\ ...,\ r_n] \in \mathcal{R}^{n\times n} \tag{18}$$

$$L = [l_1,\ l_2,\ ...,\ l_n] \in \mathcal{R}^{n\times n} \tag{19}$$

and

$$D_\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, ..., \lambda_n] \in \mathcal{R}^{n\times n} \tag{20}$$

For the general case, where the modal matrix D_Λ is of a Jordan form, the following results are also similar. Here, the continuous system case is considered. For the discrete system case, the following results can also be applied with some slight modifications. To yield a solution to the state equations a transformation is performed so that the closed-loop matrix $A + BKC$ takes on the diagonal form as given by Eq. (20). Let

$$x = Rz \tag{21}$$

where the vector $z \in \mathcal{R}^n$ is a new variable vector. Applying this transformation to the system gives

$$\dot{z} = R^{-1}(A + BKC)Rz \tag{22}$$

$$y = CRz \tag{23}$$

It is clear from the relationship between the eigenvalues and eigenvectors that

$$R^{-1}(A + BKC)R = D_\Lambda \tag{24}$$

Thus, the solution to Eqs. (22) and (23) is

$$y = CRe^{D_\Lambda t} R^{-1} x(0) \tag{25}$$

$$= CRe^{D_\Lambda t} L^T x(0) \tag{26}$$

since $R^{-1} = L^T$. This may be written as

$$y = \sum_{i=1}^{n} Cr_i e^{\lambda_i t} l_i^T x(0) \tag{27}$$

The eigenstructure plays a key role in the response of the system. It can be seen from Eq. (27) that the transient response of the system is characterized by eigenvalues together with the right and left eigenvectors. The eigenvalues determine the decay (or growth) rate of the response. The right eigenvectors fix the shape of the response. The product of the initial condition $x(0)$ and the left eigenvectors determines the amount each mode $r_i e^{\lambda_i t}$ is excited in the response. A judicious choice of a left eigenvector could prevent a mode from being excited by a known structure for the initial condition vector by choosing l_i such that $l_i^T x(0) = 0$.

Now, let us see the role of system eigenstructure assignment in the forced response of the system. The controller is then given by

$$u = K(y - r) \tag{28}$$

where r is the reference input or desired output. In a similar manner to the description, the forced response of the system is given by

$$y(t) = CRe^{D_\Lambda t} L^T x(0) + CR\int_0^t e^{D_\Lambda (t-\tau)} L^T Br(\tau)\, d\tau \tag{29}$$

The eigenstructure also plays an important role in the forced response of the system, which is the second term of Eq. (29). From Eq. (29), it is clear that the term $L^T B$ is significant in the response of the system to the reference-input r. In fact, the product $L^T B$ indicates how much a particular input excites certain modes. It may be important that a certain input has little (or ideally, no) effect on specific modes of the system. For example, in the control of an aircraft, it is desirable that a reference input surface that is used to influence the longitudinal motion (*e.g.*, the elevator) should not excite modes (*e.g.*, the dutch roll) that correspond to lateral motions of the aircraft, and vice versa. This process is termed modal decoupling. Hence, assignment of the left eigenvectors or the design of a suitable feedforward matrix needs to be considered in the design of a control system.

Therefore, in order to provide effective shaping of the response of a system, both left and right eigenvector assignment, in addition to eigenvalue assignment (or pole placement), must certainly be considered together.

4. Freedom for Eigenstructure Assignment

The freedom for the eigenstructure assignment has been examined for both the state feedback and output feedback problems. The freedom available for the eigenstructure assignment design can be stated by the following

Using the control law $u = Ky$, $\max(m, s)$ *eigenvalues may be assigned and* $\min(m, s)$ *entries of each corresponding eigenvector can be chosen precisely.*

If some freedom for eigenstructure assignment is sacrificed so that certain orthogonality conditions between left and right closed-loop eigenvectors are satisfied, it is possible to assign more than $\max(m, s)$ eigenvalues to the closed-loop system. This is termed the output feedback eigenvalue (or pole) placement problem.

To assign more than $\min(m, s)$ entries of an eigenvector, a best-fit allowable eigenvector must be chosen which approximates the desired vector according to a certain criterion. This is achieved by projecting the desired eigenvector into an allowable eigenvector subspace, which is a function of the state and input matrices, along with the specific choice of closed-loop eigenvalues.

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Biographical Sketches

Dr G. P. Liu took up the senior lecturer position in control at the University of Nottingham in May of 2000. Now he is principal lecturer in the University of Glamorgan. He received his B.Eng and M.Eng degrees in electrical and electronic engineering from the Central South University of Technology (now the Central South University) in China in 1982 and 1985, respectively, and his Ph.D. in control engineering from the University of Manchester Institute of Science and Technology (UMIST) in the UK in 1992. Dr Liu was appointed as a lecturer in the Department of Automatic Control Engineering at the Central South University of Technology in 1985. He did his postdoctoral research in the Department of Electronics at the University of York in 1992 and 1993. He worked as a research associate in the Department of Automatic Control and Systems Engineering at the University of Sheffield in 1994. During 1996-2000, he was a senior engineer in GEC-Alsthom and ALSTOM, and a principal engineer and a project leader in ABB ALSTOM Power. He has been a guest professor of the Central South University since 1994 and a visiting professor of the Institute of Automation, at the Chinese Academy of Sciences since 2000. He is a senior IEEE member. Dr Liu was awarded the Alexander von Humboldt Research Fellowship in 1992. He received the best paper prize for applications at the UKACC Conference CONTROL '98 in 1998. His paper was short listed for the best application prize at the 14th IFAC World Congress in 1999. He has more than 200 publications on control systems. He authored or co-authored 6 books: *Multiobjective Optimisation and Control*, Research Studies Press (2002); *Nonlinear Identification and Control: A Neural Network Approach*, Springer-Verlag (2001); *Eigenstructure Assignment Toolbox for Use with MATLAB*, Pacilantic International (1999); *Eigenstructure Assignment for Control System Design*, John Wiley & Sons (1998); *Advanced Adaptive Control*, Pergamon Press (1995); and *Critical Control Systems: Theory, Design and Applications*, in Research Studies Press and John Wiley & Sons (1993). He developed an eigenstructure assignment toolbox for control system design, which has been sold to many universities and companies throughout the world.

Professor Ron Patton was born in Peru, South America on March 8, 1949 and was educated at the Emmanuel Grammar School, Swansea and Sheffield University, graduating with three degrees in Electrical and Electronic Engineering and Control Systems. He has worked in the hospital service in medical physics and was a founding member of the Electronics Laboratory at the Royal Free Hospital, London in 1967/1968. During 1973/1974 he worked at the BBC Research Department, Kingswood Warren, Surrey. After completing his Ph.D. studies in 1976 he worked for GEC Electrical Projects, Rugby and Sheffield City Polytechnic on *dynamic ship positioning control systems*. He became a lecturer at Sheffield Hallam University in 1978 and moved to the new Electronics Department at York in 1981 where he focussed on fault diagnosis and aerospace control systems, with promotion to Senior Lecturer in 1987. In 1995 he was appointed Professor of Control and Intelligent Systems Engineering at Hull where his research interests are fault-tolerant control and the use of AI/soft computing methods for fault diagnosis in control systems. Ron is subject editor in fault-tolerant control, supervision and diagnosis for the International Journal of Adaptive Control and Signal processing. He chairs the IFAC Technical Committee *SAFEPROCESS*. He has published more than 250 papers and 5 books on eigenstructure assignment for control systems design, fault diagnosis and fault-tolerant control.