LYAPUNOV STABILITY

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Contents

Summary

This chapter presents the basic concepts and theorems of Lyapunov's method for studying the stability of nonlinear systems, including the invariance principle and the linearization method.

1. Introduction

Stability theory plays a central role in control theory and engineering. There are different kinds of stability problems that arise in the study of dynamical systems (see Stability theory, Popov and circle criterion, and Input-output stability). This article is concerned with Lyapunov stability. Stability of equilibrium points is defined in Section 2 for autonomous systems and Lyapunov's theorem is given.

An extension of the basic theory, known as the invariance principle, is given in Section 3. For a linear time-invariant system $\dot{x}(t) = Ax(t)$, stability of the equilibrium point $x = 0$ can be completely characterized by the location of the eigenvalues of A. This is discussed in Section 4. In Section 5, it is shown when and how the stability of an equilibrium point can be determined by linearization about that point.

In Section 6, we extend Lyapunov's method to non-autonomous systems. We define the concepts of uniform stability, uniform asymptotic stability, and exponential stability of the equilibrium point of a non-autonomous system and give Lyapunov's method for testing them.

2. Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{1}$$

where $f : D \to R^n$ is a locally Lipschitz map from a domain $D \subset R^n$ into R^n. Suppose $\bar{x} \in D$ is an equilibrium point of (1); that is, $f(\bar{x}) = 0$. Our goal is to characterize and study stability of $\bar{x}$.

For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin $x = 0$. There is no loss of generality in doing so because any equilibrium point $\bar{x}$ can be shifted to the origin via the change of variables $y = x - \bar{x}$.

Definition 1 *The equilibrium point* $x = 0$ *of* (1) *is stable if, for each* $\varepsilon > 0$, *there is* $\delta = \delta(\varepsilon) > 0$ *such that* $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon$ *for all* $t \ge 0$; *unstable if not stable; asymptotically stable if it is stable and* δ *can be chosen such that* $\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0$.

Let $V : D \to R$ be a continuously differentiable function defined in a domain $D \subset R^n$ that contains the origin. The derivative of V along the trajectories of (1), denoted by $\dot{V}(x)$, is given by

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ \ldots, \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \frac{\partial V}{\partial x} f(x)$$

If $\dot{V}(x)$ is negative, V will decrease along the trajectory of (1) passing through x. A function $V(x)$ is *positive definite* if $V(0) = 0$ and $V(x) > 0$ for $x \ne 0$. It is *positive semidefinite* if it satisfies the weaker condition $V(x) \ge 0$ for $x \ne 0$. A function $V(x)$ is *negative definite* or *negative semidefinite* if $-V(x)$ is positive definite or positive semidefinite, respectively.

Lyapunov's stability theorem states that *the origin is stable if, in a domain* D *that contains the origin, there is a continuously differentiable positive definite function* $V(x)$ *so that* $\dot{V}(x)$ *is negative semidefinite, and it is asymptotically stable if* $\dot{V}(x)$ *is negative definite.* When the condition for stability is satisfied, the function V is called a Lyapunov function. The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov surface* or a *level surface*. Using Lyapunov surfaces, Figure 1 makes the theorem intuitively clear. It shows Lyapunov surfaces for increasing values of c. The condition $\dot{V} \le 0$ implies that when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{V(x) \le c\}$ and can never come out again. When $\dot{V} < 0$, the trajectory moves from one Lyapunov surface to an inner Lyapunov surface with a smaller c. As c decreases, the Lyapunov surface $V(x) = c$ shrinks to the origin, showing that the trajectory approaches the origin as time progresses.

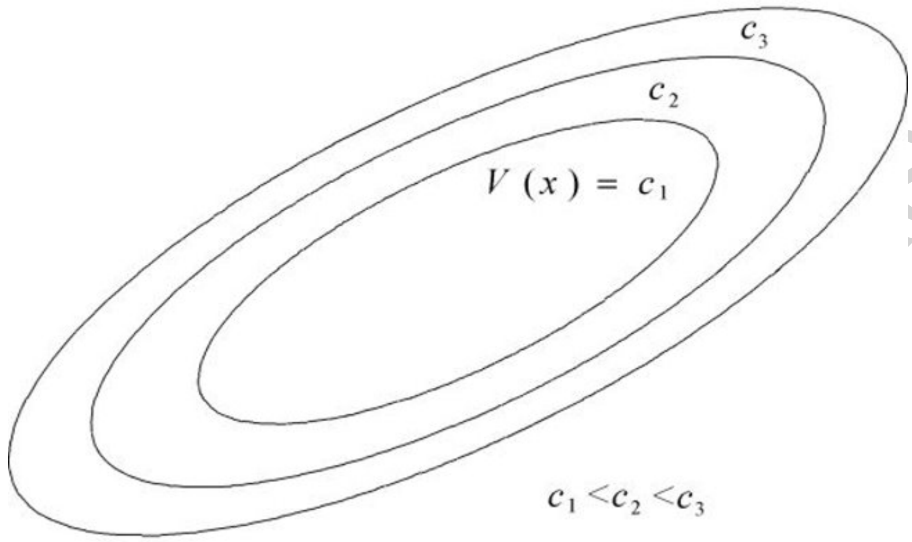

Figure 1: Level surfaces of a Lyapunov function.

Lyapunov's theorem can be applied without solving the differential equation (1). On the other hand, there is no systematic method for finding Lyapunov functions. In some cases, there are natural Lyapunov function candidates like energy functions in electrical or mechanical systems. In other cases, it is basically a matter of trial and error.

Example 1 Consider the first-order differential equation $\dot{x} = -g(x)$ where $g(x)$ is locally Lipschitz on $D = (-a, a)$, $g(0) = 0$, and $xg(x) > 0$, $\forall x \in D - \{0\}$. Over the domain D, $V(x) = \int_0^x g(y)\, dy$ is continuously differentiable, $V(0) = 0$, and $V(x) > 0$ for all $x \neq 0$. Thus, $V(x)$ is a valid Lyapunov function candidate. To see whether or not $V(x)$ is indeed a Lyapunov function, we calculate its derivative along the trajectories of the system.

$$\dot{V}(x) = \frac{\partial V}{\partial x}\left[-g(x)\right] = -g^2(x) < 0, \forall x \in D - \{0\}$$

Thus, the origin is asymptotically stable.

Example 2 Consider the pendulum equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a\sin x_1 - bx_2, \quad a > 0, b \geq 0$$

and let us study stability of the equilibrium point at the origin. A natural Lyapunov function candidate is the energy function $V(x) = a(1-\cos x_1) + (1/2)x_2^2$. Clearly, $V(0) = 0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = a\,\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 - bx_2^2 = -bx_2^2$$

Thus, the origin is stable. When $b = 0$ (no friction), $\dot{V}(x) \equiv 0$. In this case, we can conclude that the origin is not asymptotically stable; for trajectories starting on a Lyapunov surface $V(x) = c$ remain on the same surface for all future time. When $b > 0$, $\dot{V}(x) = -bx_2^2$ is negative semidefinite, but not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1; that is, $\dot{V}(x) = 0$ along the x_1-axis. Therefore, we can only conclude that the origin is stable. We will see in the next section that the origin is asymptotically stable.

This example emphasizes an important feature of Liapunov's stability theorem; namely, *the theorem's conditions are only sufficient.* Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that he equilibrium is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate. Whether the equilibrium point is stable (asymptotically stable) or not can be determined only by further investigation.

When the origin is asymptotically stable, we are often interested in determining how far from the origin the trajectory can be and still converge to the origin as t approaches ∞. This gives rise to the definition of the *region of attraction* (also called *region of asymptotic stability*, *domain of attraction*, or *basin*) as the set of all points x_0 such that the solution of $\dot{x} = f(x)$, $x = (0) = x_0$, tends to zero at $t \to \infty$. Finding the exact region of attraction analytically might be difficult or even impossible. However, Lyapunov functions can be used to estimate the region of attraction, that is, to find sets contained in the region of attraction. If there is a Lyapunov function that satisfies the conditions of asymptotic stability over a domain D, and if $\Omega_c = \{V(x) \leq c\}$ is bounded and contained in D, then every trajectory starting in Ω_c remains in Ω_c and approaches the origin as $t \to \infty$. Thus, Ω_c is an estimate of the region of attraction. This estimate, however, may be conservative; that is, it may be much smaller that the actual region of attraction. The origin is *globally asymptotically stable* if the region of attraction is the whole space R^n. The Barbashin-Krasovskii theorem states that *the origin is globally asymptotically* *stable if* $V(x)$ *is positive definite and* $\dot{V}$ *is negative definite for all* $x \in R^n$ *and if, in addition,* $V(x)$ *is radially unbounded; that is,* $V(x) \to \infty$ *as* $\|x\| \to \infty$. The radial unboundedness condition guarantees that the set Ω_c will be bounded for any $c > 0$; hence, any initial state can be included in Ω_c by choosing c large enough.

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Biographical Sketch

Hassan K. Khalil received the B.S. and M.S. degrees from Cairo University, Cairo, Egypt, and the Ph.D. degree from the University of Illinois, Urbana-Champaign, in 1973, 1975, and 1978, respectively, all in Electrical Engineering.

Since 1978, he has been with Michigan State University, East Lansing, where he is currently Professor of Electrical and Computer Engineering. He has consulted for General Motors and Delco Products.

He has published several papers on singular perturbation methods, decentralized control, robustness, nonlinear control, and adaptive control. He is author of the book Nonlinear Systems (Macmillan, 1992; Prentice Hall, 1996 and 2002), coauthor, with P. Kokotovic and J. O'Reilly, of the book Singular Perturbation Methods in Control: Analysis and Design (Academic Press, 1986; SIAM 1999), and coeditor, with P. Kokotovic, of the book Singular Perturbation in Systems and Control (IEEE Press, 1986). He was the recipient of the 1983 Michigan State University Teacher Scholar Award, the 1989 George S. Axelby Outstanding Paper Award of the IEEE Transactions on Automatic Control, the 1994 Michigan State University Withrow Distinguished Scholar Award, the 1995 Michigan State University Distinguished Faculty Award, the 2000 American Automatic Control Council Ragazzini Education Award, and the 2002 IFAC Control Engineering Textbook Prize. He is an IEEE Fellow since 1989.

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