STABILITY OF HYBRID SYSTEMS

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Contents

Summary

This section collects work on the general stability analysis of hybrid systems. The hybrid systems considered are those that combine continuous dynamics—represented by differential or difference equations-with finite dynamics—usually thought of as being a finite automaton. We present some general background on stability analysis, and then work on the stability analysis of hybrid control systems. Specifically, we review *multiple Lyapunov functions* as a tool for analyzing Lyapunov stability. Other stability notions and other analysis tools are discussed in the subsection Going Further. Specializing to hybrid systems with linear dynamics in each constituent mode and linear jump operators, we review some key theorems for impulsive systems and give corollaries encompassing several recently-derived "stability by first approximation" theorems in the literature.

1. Background and Motivation

Suppose we are given a dynamical system in R^n specified by a differential (respectively, difference) equation:

$$\Sigma:\ \dot{x}(t) = f\left(x(t)\right) \qquad \left(\text{respectively, } x(t+1) = f\left(x(t)\right)\right). \tag{1}$$

An important concept when analyzing such systems is stability. We give a taste of stability theory below and refer the reader to the excellent introduction by Luenberger for further details and more advanced references.

Stability means that small changes in operating conditions, such as differences in initial data, lead to small changes in behavior. Specifically, let $x(t)$ and $z(t)$ be solutions of Σ when the initial conditions are x_0 and z_0, respectively. Further, let $\|\cdot\|$ denote the Euclidean distance between vectors, $\|z-x\| = \left[\sum_{i=1}^{n}(z_i - x_i)^2\right]^{1/2}$.

Definition 1 (Lyapunov Stability of Solution) *A solution* $x(t)$ *of* $\sum$ *is Lyapunov stable if for any* $\mathcal{E} > 0$, *there exists a* $\delta(\mathcal{E}) > 0$ *such that all solutions of* $\sum$ *with* $\|z(0) - x(0)\| < \delta$ *are such that* $\|z(t) - x(t)\| < \mathcal{E}$ *for all* $t > 0$. Δ

An *equilibrium point* of $\sum$ is one which remains unchanged under the dynamics, namely a point $\bar{x}$ where $f(\bar{x}) = 0$ (respectively, $f(\bar{x}) = \bar{x}$). Since the equilibrium point $\bar{x}$ is a particular kind of solution (namely, one where $x(t) \equiv \bar{x}$ for all *t*), we may talk of the Lyapunov stability of equilibrium points. Instead of simply repeating the definition for this specific case, though, it is convenient to introduce some new notation. Let $B(\bar{x}, R)$ denote the ball of radius *R* about $\bar{x}$, that is, all the points *y* in the state space such that $\|\bar{x} - y\| < R$.

Definition 2 (Lyapunov Stability of Equilibrium Point) *An equilibrium point* $\bar{x}$ *of* $\sum$ *is Lyapunov stable if for any* $R > 0$, *there exists an r*, $0 < r < R$, *such that if* z_0 *is inside* $B(\bar{x}, r)$, *then* $z(t)$ *is inside* $B(\bar{x}, R)$ *for all t* > 0. Δ

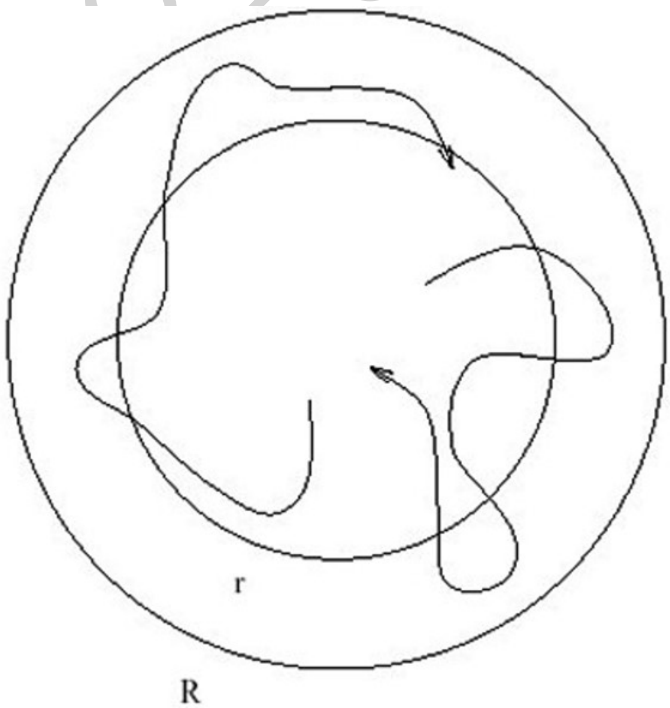

Figure 1: Lyapunov stability

For all illustration of the concept, see Figure 1, which depicts two stable trajectories in continuous time. Other notions are defined in terms of this primitive concept. For instance, an equilibrium point is *asymptotically stable* if it is stable and there is a $A > 0$ such that if the system is initiated inside $B(\bar{x}, A)$ the trajectory is attracted to $\bar{x}$ as time increases; it is exponentially stable if it is attracted to $\bar{x}$ at an exponential rate, i.e., $\|x(t) - \bar{x}\| \le ce^{-\mu t}, c, \mu > 0$; it is *globally asymptotically stable* if *A* may be taken arbitrarily large; it is *unstable* if it is not stable.

Besides introducing the notion of stability above, Lyapunov devised two methods for testing the stability of an equilibrium point, which have come to be known as (1) Lyapunov's indirect method, and (2) Lyapunov's direct method. The indirect method involves examining the stability of a linearized version of the function *f*. Specially, one examines the *Jacobian matrix*, the $n \times n$ matrix of first derivatives of *f* with respect to *x*, evaluated at the equilibrium point:

$$F = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=\bar{x}} \tag{2}$$

First order approximation near the equilibrium point,

$$f(\bar{x} + y) \approx f(\bar{x}) + Fy = Fy, \qquad (\text{respectively, } \bar{x} + Fy) \tag{3}$$

so that

$$\dot{y}(t) = Fy(t) \qquad (\text{respectively, } y(t+1) = Fy(t)) \tag{4}$$

gives a linear approximation of the perturbations to the solution of ∑ near $\bar{x}$. In some cases, the stability properties of the system in ∑ can be inferred from those of the linear system in Eq.(4). In particular, if all the eigenvalues of *F* have strictly negative real parts (respectively, magnitude strictly less than one), then $\bar{x}$ is an asymptotically stable equilibrium point of ∑. If any eigenvalue has a positive real part (respectively, magnitude greater than one), it is unstable. If all have non-positive real parts but some have zero real parts (respectively, have magnitude less than or equal to one but some have unity magnitude), then nothing can be concluded about stability from this indirect method alone.

Lyapunov's other, direct method for verifying stability works directly with the nonlinear system rather than its linearized version. The basic idea is to seek a type of "energy function" that "decreases along trajectories of the system." Next, we make these notions precise. Suppose that $\bar{x}$ is an equilibrium point of a given dynamic system.

Definition 3 (Lyapunov Function) A candidate Lyapunov function *for the system* $\sum$ *and the equilibrium point* $\bar{x}$ *is a real-valued function V, which is defined over a region* Ω *of the state space that contains* $\bar{x}$, *and satisfies the two requirements:*

- Continuity. *V is continuous and, in the case of a continuous-time system, V has continuous derivative*.
- Positive Definiteness. $V(x)$ *has a unique minimum at* $\bar{x}$ *with respect to all other points in* Ω. *Without loss of generality, we henceforth assume* $V(\bar{x}) = 0$.

A Lyapunov function *for the system* $\sum$ *and the equilibrium point* $\bar{x}$ *is a candidate Lyapunov function V which also satisfies the requirement*.

- [Non-increasing. *Along any trajectory of the system contained in* Ω *the value of V never increases*. *That is, for a continuous-time system, the function* $\dot{V}(x) = \nabla V(x) f(x) \le 0$ *for all x in* Ω, ($\nabla V(x)$ is the *gradient vector*) *for a discrete-time system, the function* $\Delta V(x) = V(f(x)) - V(x) \le 0$ *for all x in* Ω.

$$\left[\frac{\partial V(x)}{\partial x_1}, \frac{\partial V(x)}{\partial x_2}, \ldots, \frac{\partial V(x)}{\partial x_n}\right] \tag{5}$$

With these definitions, we may state the following important theorem (see Luenberger's book, or our Theorem 16 in the case *N* = 1, for continuous- and discrete-time proofs).

Theorem 4 (Lyapunov Theorem) *If there exists a Lyapunov function* $V(x)$ *in the region* $B(\bar{x}, R)$, $R > 0$, *then the equilibrium point* $\bar{x}$ *is Lyapunov stable*.

Summarizing, to use Lyapunov's direct method (in continuous time, for example) you

1. After examining your system, pick a Lyapunov function candidate *V*;
2. Compute $\dot{V}$ (respectively, ΔV);
3. Draw conclusions about the system $\sum$ in Eq.(1).

See Figure 2, which depicts these steps. Also, some important things to note are

- Engineering insight is used to pick *V*, e.g., in mechanical and electrical problems, *V* can often be chosen as the total (Kinetic plus potential) energy of the system.
- The above Lyapunov theorem has a converse, but its sufficiency form stated above is often useful as a design tool (e.g., in adaptive control, where one chooses a candidate Lyapunov function and then a parameter update rule that will result in its being non-increasing over trajectories.

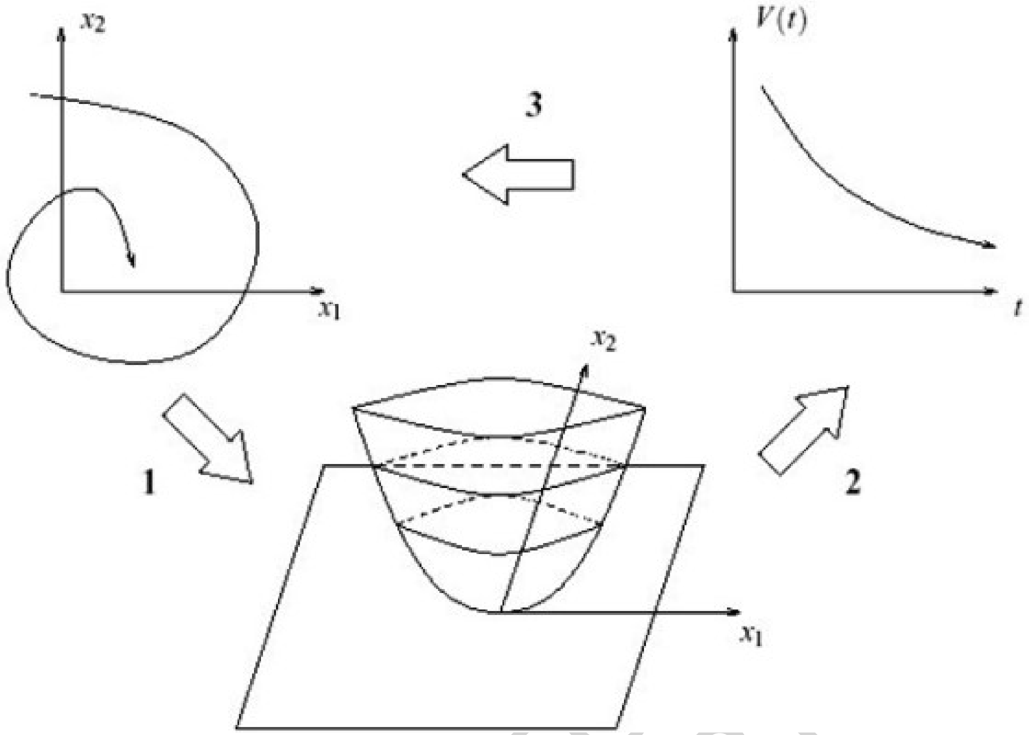

Figure 2: Systems analysis cycle for Lyapunov's direct method.

Given the above, whole books may be written—and many have—on the qualitative theory of dynamical systems, extending to theorems on asymptotic stability and instability, global and uniform versions, etc. For one example,

Theorem 5 (Luenberger) *Suppose V is a Lyapunov function for a dynamic system and an equilibrium point* $\bar{x}$ *. Suppose in addition that*

- *V is defined on the entire state space.*
- $\dot{V}(x) < 0$ (*respectively,* $\Delta V(x) < 0$) *for all* $x \neq \bar{x}$.
- *V*(*x*) *goes to infinity as* $\|x - \bar{x}\|$ *goes to infinity*.

Then $\bar{x}$ *is globally asymptotically stable*.

1.1. What is a Hybrid System?

The simplest hybrid system is a *switched system*:

$$\dot{x}(t) = f_q(x(t)), \quad q \in \{1, \ldots, N\}, \tag{6}$$

where $x(t) \in R^n$. We add the following assumptions. (1) Each f_q is globally Lipschitz continuous. (2) The *q's* are picked in such a way that there are finite switches in finite time.

Such systems are of “variable structure” or “multi-modal”; they are a simple model of (the continuous portion) of hybrid systems. We explain this below. The particular q at any given time may be chosen by some “higher process,” such as a controller, computer, or human operator, in which case we say that the system is *controlled*. It may also by a function of time or state or both, in which case we say that the system is *autonomous*. In the latter case, we may really just arrive at a single (albeit complicated) nonlinear, time-varying equation. However, one might gain some leverage in the analysis of such systems by considering them to be amalgams of simpler systems.

A real-world example of a switched system is one that arises in the control of the longitudinal dynamics of an aircraft. See Figure 3. It is desired that there is good tracking of pilot’s input, n_z, without violating angle-of-attack constraint. To accomplish this, engineers build a good tracking controller and a good safety control (which regulates about the maximum angle of attack) and combine them using simple logic. The resulting Max Controller is shown in Figure 4. It achieves the stated objectives.

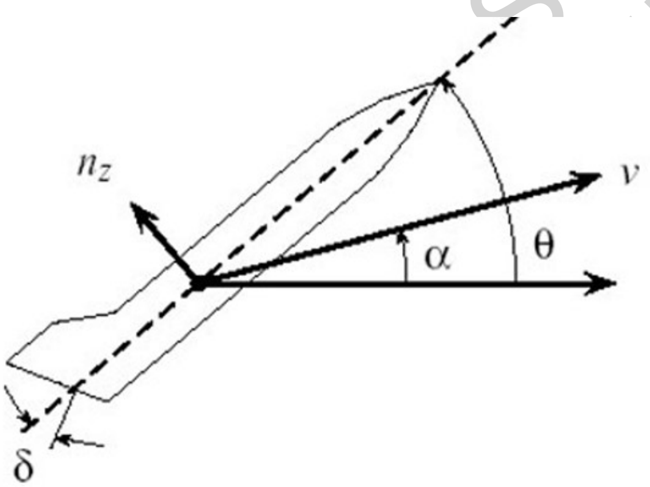

Figure 3: Longitudinal Aircraft View.

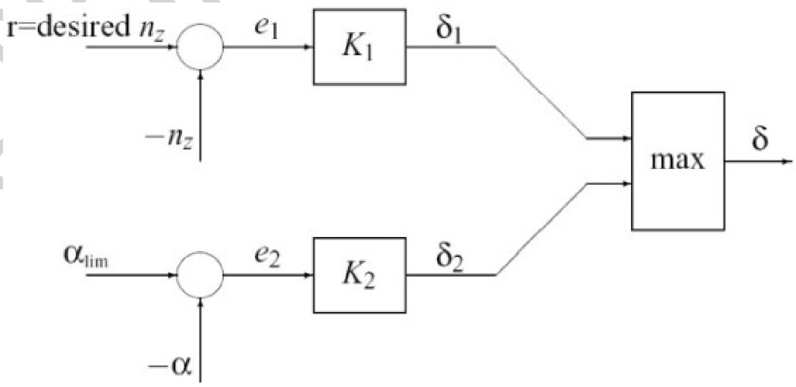

Figure 4: The Max Controller.

A particular case of interest for Eq.(6) is the case of *switched linear systems*, where each of the f_q is a linear system:

$$\dot{x}(t) = A_q\left(x(t)\right), \qquad q \in \{1, \ldots, N\}, \tag{7}$$

where $x(t) \in R^n$.

In addition to the switching phenomenon discussed above, so-called *systems with impulse effect* often add the possibility of the state's jumping (also known as "resets") when certain boundaries are crossed. In general, these boundaries are subsets of the space, M, but they may be given explicit representation in terms of the zeros of one or more functions.

$$\dot{z}(t) \quad = \quad f\left(z(t)\right), \qquad (z,t) \notin M_t, \tag{8}$$

$$z\left(t^+\right) \quad = \quad J\left(z(t)\right), \qquad (z,t) \in M_t \tag{9}$$

The interpretation of the above is that the dynamics evolves according to the differential equation while (z,t) is in the complement of $M_t \subset Z \times I$, but that the state is immediately reset according to the map J upon the (z,t)'s hitting the set M_t. See Bainov and Simeonov's book and Branicky's thesis for more details and conditions on when the dynamics is well-defined. There are three main cases of interest:

- Fixed Instants of Impulse Effect. The sets M_t are hyperplanes at fixed instants of $t = \tau_1, \tau_2, \ldots.$
- Mobile Instants of Impulse Effect. The sets M_t are a sequence of hypersurfaces $\sigma_k = \tau_k(x)$.
- Autonomous Impulse Effect. The sets M_t are constraints on the state space, i.e., they are of the form $M \times I, M \subset Z$.

As an example of a hybrid system consider Pait's two-state stabilizer for the simple harmonic oscillator (S.H.O.). See Figure 5. Here, the hybrid state consists of values for the continuous variables x and y, plus a location (discrete state) in the automaton.

The dynamics of this flavor of hybrid system are such that it stays in a location for T seconds (as defined in that location) and then follows the transition arrow which is active (determined by conditions on the values of the continuous variables) to the next location.

The system begins in State 2 (left) and stays there for $3\pi/4$ seconds. Example dynamics are shown in Figure 6.

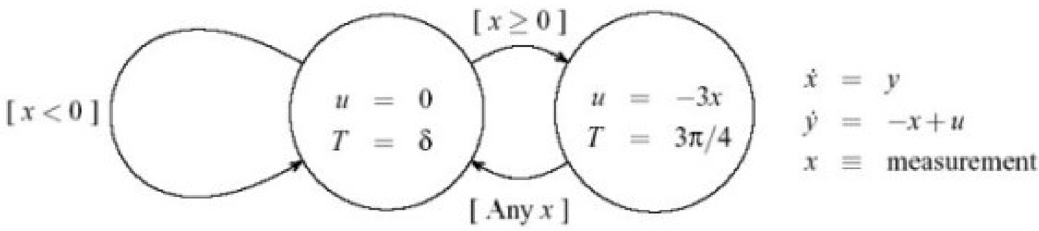

Figure 5: Pait’s two-state hybrid S.H.O. stabilizer and S.H.O. equations.

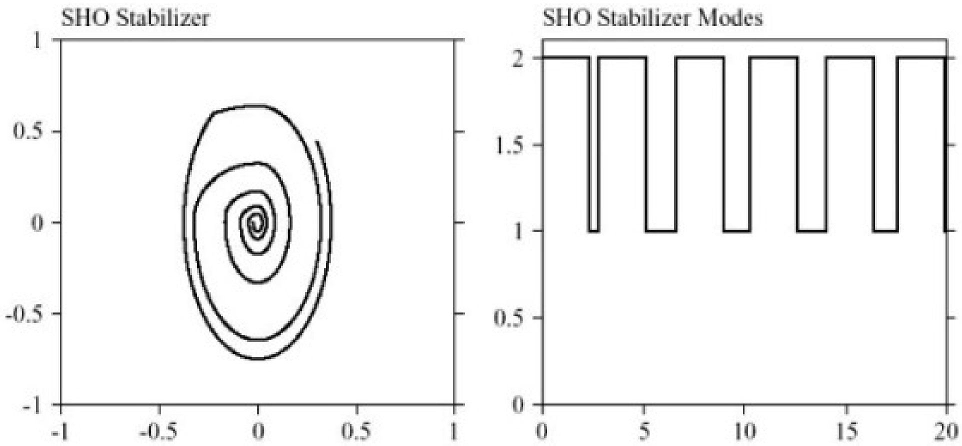

Figure 6: Phase Portrait and Mode Switchings for Pait’s S.H.O. Stabilizer

A *hybrid dynamical system* is simply an indexed collection of dynamical systems plus rules for switching among them “jumping” among them (switching dynamical system and/or resetting the state). See Figure 7. This jumping occurs whenever the state satisfies certain conditions, given by its membership in a specified subset of the state space. Hence, the entire system can be thought of as a sequential patching together of dynamical systems with initial and final states, the jumps performing a reset to a (generally different) initial state of a (generally different) dynamical system whenever a final state is reached.

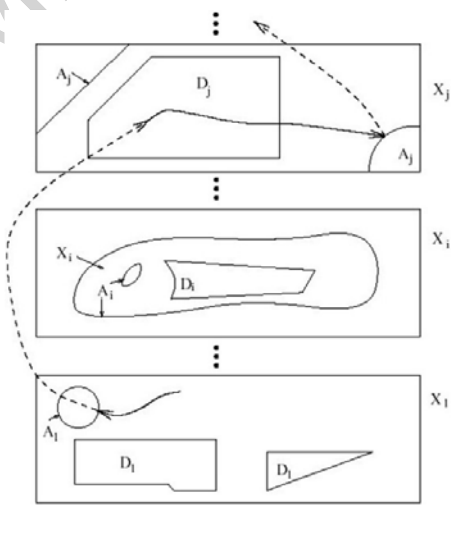

Figure 7: Example hybrid dynamical system.

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Biographical Sketch

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