DISCRETE-TIME EQUIVALENTS TO CONTINUOUS-TIME SYSTEMS

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Contents

Summary

When a digital controller is designed to control a continuous-time plant it is important to have a good understanding of the plant to be controlled as well as of the controller and its interfaces with the plant. There are two fundamental approaches to designing discrete-time control systems for continuous-time plants.

The first approach is to derive a discrete-time equivalent of the plant and then design a discrete-time controller directly to control the discretized plant. This approach to

designing a digital controller directly, which has many variations, parallels the classical approach to analog controller design. One begins with simple discrete-time controllers, increasing their complexity until both steady state error and transient performance requirements are met.

The other approach to designing discrete-time control systems for continuous-time plants is to first design a continuous-time controller for the plant, then derive a digital filter that closely approximates the behavior of the original analog controller.

The filter design can approximate the integrations with discrete-time operations or it can be made to have step (or other) response samples that are equal to samples of the analog controller's step (or other) response. Usually, however, even for small sampling periods, the discrete-time approximation performs less well than the continuous-time controller from which it was derived.

In this chapter, several classical and state space methods for discretizing continuous-time systems are developed and illustrated.

1. Introduction

The rapid development of digital technology continues to change the boundaries of control system design options. It is now routinely feasible to implement very complicated digital controllers and perform the extensive calculations required for their design. These advances in implementation and design capability can be achieved at low cost because of the widespread availability of inexpensive, powerful digital computers and related devices.

A digital control system uses digital hardware, usually in the form of a programmed digital computer as the heart of the controller. In contrast, the controller in an analog control system uses analog electronics, mechanical, electromechanical or hydraulic devices. Digital controllers normally have analog elements at their periphery to interface with the plant; it is the internal workings of the controller that distinguishes digital from analog control.

An example of a digital control system for a continuous-time plant is shown in Figure 1. The system has two reference inputs and four outputs, only two of which are measured by the two sensors. The analog-to-digital converters (A/D) perform sampling of the sensor signals and produce binary representations of these sensors signals.

The digital controller algorithm in the digital computer then modifies sensor signals and generates control inputs $u_1(k)$ and $u_2(k)$. The control inputs $u_1(k)$ and $u_2(k)$ are then converted to analog signals via digital-to-analog converters (D/A). The analog signals $u_1(t)$ and $u_2(t)$ are applied to the plant actuators or control elements to control the behavior of the plant.

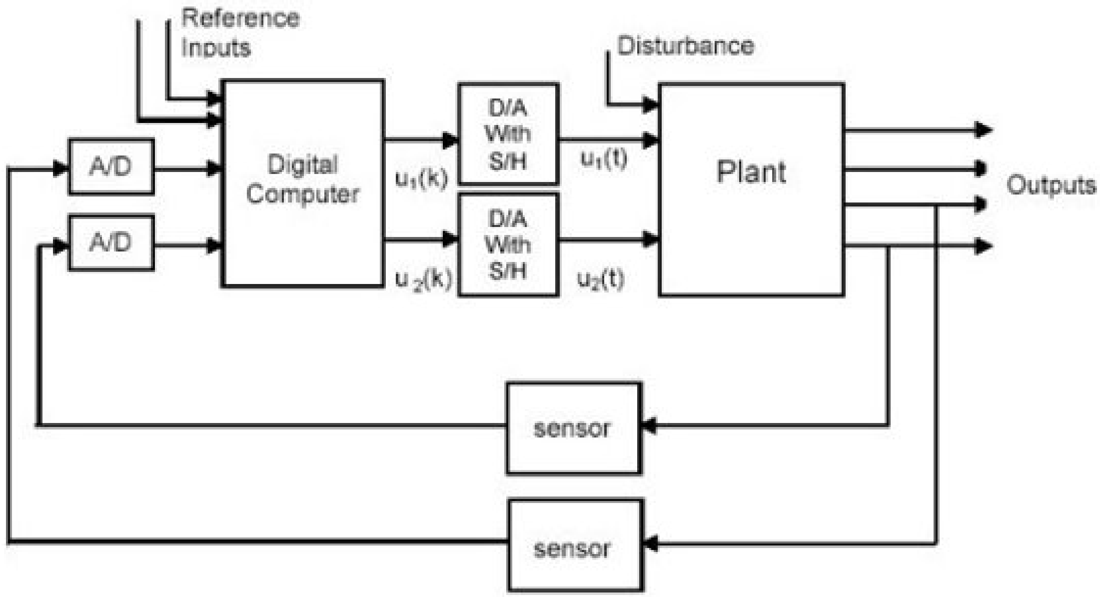


Figure 1: A Digital Control System Controlling a Continuous-Time Plant

2. Design of Discrete-Time Control Systems for Continuous-Time plants

There are two fundamental approaches to designing discrete-time control systems for continuous-time plants. The first approach is to derive a discrete-time equivalent of the plant and then design a discrete-time controller directly to control the discretized plant. This approach is discussed in section 3. The other and more traditional approach to designing discrete-time control systems for continuous-time plants is to first design a continuous-time controller for the plant, then derive a discrete-time equivalent that closely approximates the behavior of the original analog controller. This approach is especially useful when an existing continuous-time controller or a part of the controller is to be replaced with a discrete-time controller. Usually, however, even for small sampling periods, the discrete-time approximation performs less well than the continuous-time controller from which it was derived. The approach to deriving a discrete-time controller that closely approximates the behavior of the original analog controller is discussed in section 4.

Before we discuss discrete-time equivalents of continuous-time systems, it is instructive to briefly discuss sampling and reconstruction in order to gain greater insight into the process of discretizing continuous-time systems.

2.1. Sampling and A/D Conversion

Sampling is the process of deriving a discrete-time sequence from a continuous-time function. As shown in Figure 2, an incoming continuous-time signal $f(t)$ is sampled by an A/D converter to produce the discrete-time sequence $f(k)$. Usually, but not always, the samples are evenly spaced in time. The sampling interval T is generally known and is indicated on the diagram or elsewhere.

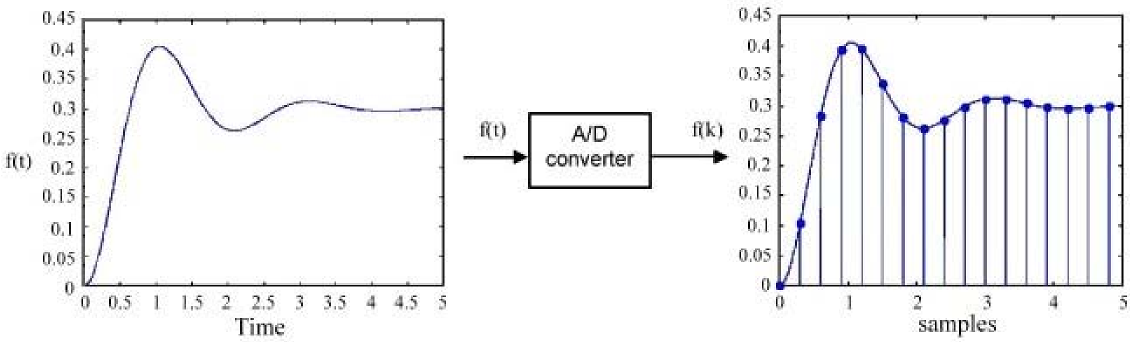


Figure 2: Sampling of a Continuous-Time Signal Using an A/D Converter

The A/D converter produces a binary representation, using a finite number of bits, of the applied input signal at each sample time. Using a finite number of bits to represent a signal sample generally results in quantization errors in the A/D process. For example, the maximum quantization error in 16-bit A/D conversion is 2^{-16} = 0.0015%, which is very low compared with typical errors in analog sensors. This error, if taken to be "noise", gives a signal-to-noise (SNR) of $20\log_{10}(2^{-16})$ = 96.3 db which is much better than that of most control systems.

The control system designer must ensure that enough bits are used to give the desired system accuracy. Study of the effects of roundoff or truncation errors in digital computation is beyond our scope in this chapter, but it is important to use adequate word lengths in fixed or floating point computations. Years ago, digital hardware was very expensive, so minimizing word length was much more important than it is today.

When a continuous-time signal $f(t)$ is sampled to form the sequence $f(k)$, there exists a relationship between the Laplace transform of $f(t)$ and the z-transform of $f(k)$. If a rational Laplace transform is expanded into partial fraction terms, the corresponding continuous-time signal components in the time domain are powers of time, exponentials, sinusoids, and so on.

Uniform samples of these elementary signal components have, in turn, simple z-transforms that can be summed to give the z-transform of the entire sampled signal. Table 1 lists some Laplace transform terms and the resulting z-transforms when the corresponding time functions are sampled uniformly.

$f(t)$	$F(s)$	$f(k)$	$F(z)$
$u(t)$, unit step	$\frac{1}{s}$,	$u(k)$ unit step	$\frac{z}{z-1}$
$tu(t)$	$\frac{1}{s^2}$	$kTu(k)$	$\frac{Tz}{(z-1)^2}$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$(e^{-aT})^k u(k)$	$\frac{z}{z-e^{-aT}}$

$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$kT(e^{-aT})^k u(k)$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$	$\sin(k\omega T)u(k)$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$	$\cos(k\omega T)u(k)$	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$

Table 1: Laplace and Z-Transform Pairs

As an example, consider the continuous-time function with Laplace transform

$$F(s)=\frac{2}{s(s+2)}=\frac{1}{s}+\frac{-1}{s+2}$$

The z-transform of the sampled signal with a sampling interval $T=0.1$ seconds is

$$F(z)=\frac{z}{z-1}-\frac{z}{z-e^{-0.2}}=\frac{0.18z}{(z-1)(z-0.82)}$$

2.2. Reconstruction and D/A Conversion

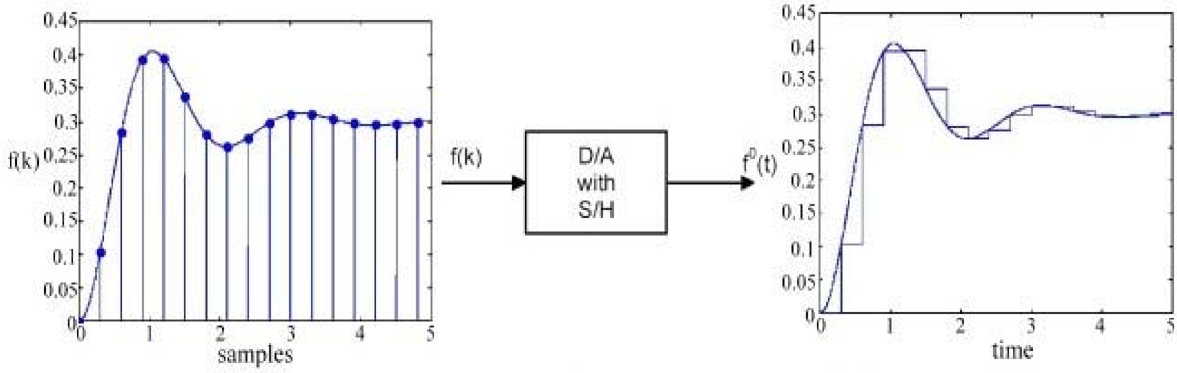
Reconstruction is the formation of a continuous-time function from a sequence of samples. Many different continuous-time functions can have the same set of samples, so a reconstruction is not unique.

Reconstruction is performed using D/A converters. Electronic D/A converters typically produce a step reconstruction from incoming signal samples by converting the binary-coded digital input to a voltage, transferring the voltage to the output, and holding the output voltage constant until the next sample is available.

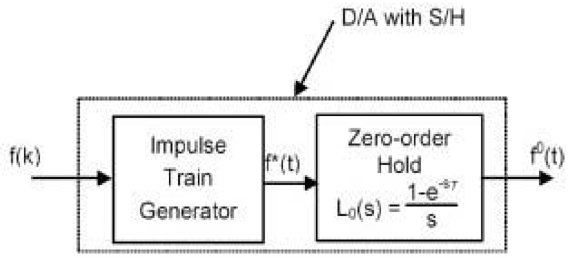
The symbol for a D/A converter that generates the step reconstruction $f^0(t)$ from signal samples $f(k)$ is shown in Figure 3(a). Sample and hold (S/H) is the operation of holding each of these samples for a sampling interval T to form the step reconstruction. As shown in Figure 3(b), the step reconstruction of a continuous-time signal from samples can be represented as the conversion of the sequence $f(k)$ to its corresponding impulse train $f^*(t)$, where

$$f^*(t)=\sum_{k=0}^{\infty} f(k)\delta(t-kT) \tag{1}$$

then conversion of the impulse train to the step reconstruction. This viewpoint neatly separates conversion of the discrete sequence to a continuous-time waveform and the details of the shape of the reconstructed waveform.



3(a) Symbol for D/A Conversion with S/H



$$f^*(t) = \sum_{k=0}^{\infty} f(k)\delta(t-kT)$$

3(b) Representations of a D/A converter with S/H

Figure 3: Digital-to-Analog (D/A) Conversion with Sample and Hold (S/H)

The continuous-time transfer function that converts the impulse train with sampling interval T to a step reconstruction is termed zero-order-hold (ZOH). Each incoming impulse in equation (1) to the ZOH produces a rectangular pulse of duration T. Therefore; the transfer function of the ZOH is given by:

$$L_o(s) = \frac{1}{s}(1 - e^{-sT})$$

One way to improve the accuracy of the reconstruction is to employ holds that are higher-order than the zero-order hold. An n^{th} order hold produces a piecewise n^{th} degree polynomial that passes through the most recent $n+1$ input samples.

It can be shown that, as the order of the hold is increased, a well-behaved signal is reconstructed with increased accuracy. For example, a first order hold (FOH) uses the previous two samples to construct a straight-line approximation during each interval. The transfer function of the FOH is:

$$L_1(s) = \frac{(Ts+1)(1 - e^{-sT})^2}{Ts^2}$$

A model of the FOH is shown in Figure 4(a). If the hardware of the FOH is not available, one can implement a FOH as shown in Figure 4(b).

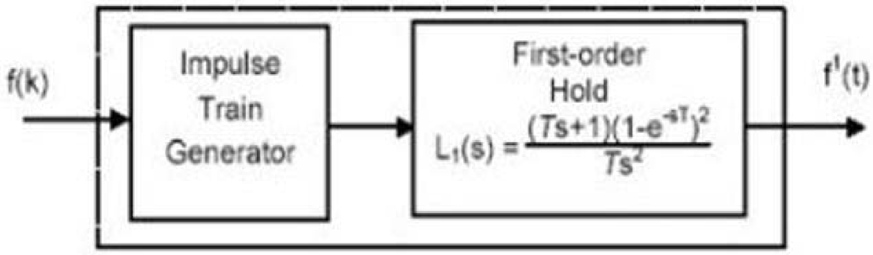


Figure 4(a)

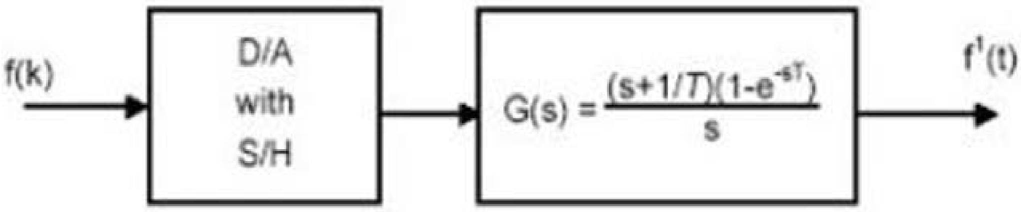


Figure 4(b)

Figure 4: First-order hold reconstruction

3. Discrete-Time Equivalents of Continuous-Time Plants

The first approach to designing discrete-time control systems for continuous-time plants is to derive a discrete-time equivalent of the plant and then design a discrete-time controller directly to control the discretized plant.

Consider the general configuration shown in Figure 5(a) where it is desired to design a discrete-time controller transfer function $G_c(z)$ to control the continuous-time plant described by the transfer function $G_p(s)$.

The first step is to derive a discrete-time equivalent of the plant described by $G_p(s)$ as shown in Figure 5(b). To do so, the dashed portion of Figure 5(b) has been redrawn in Figure 6(a) to emphasize the relationship between the discrete-time signals, $f(k)$ and $y(k)$.

It is desired now to find the discrete-time transfer function $G_p(z)$ of the arrangement, and this can be done by finding its pulse response.

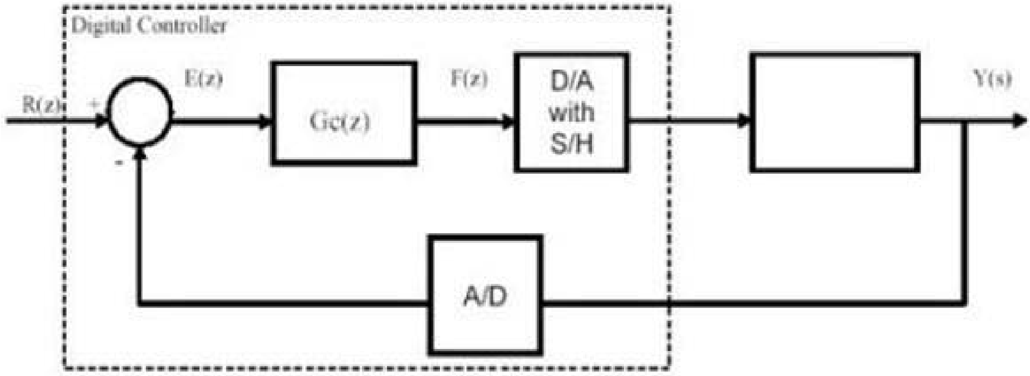


Figure 5(a)

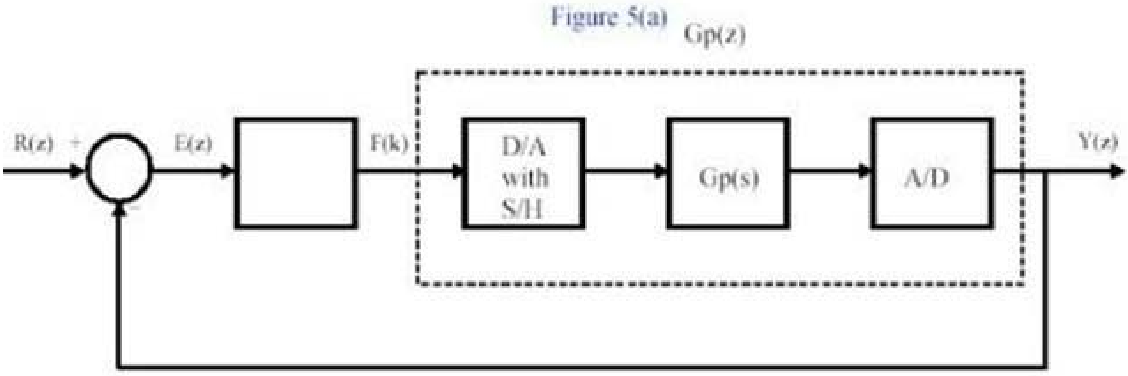


Figure 5(b)

Figure 5: A discrete-time equivalent of a continuous-time plant

For a unit pulse input of

$$f(k) = \delta(k)$$

the sampled-and-held continuous-time signal that is the input to $G_{\mathrm{p}}(s)$ is given by

$$f^{0}(t) = u(t) - u(t-T)$$

or

$$F^{0}(s) = \frac{1-e^{-sT}}{s}$$

where T is the sampling interval. Then

$$Y(s) = F^{0}(s)G_{\mathrm{p}}(s) = \frac{1-e^{-sT}}{s}G_{\mathrm{p}}(s) \tag{2}$$

and therefore

$$G_{\mathrm{p}}(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s} G_{\mathrm{p}}(s)\right] \quad (3)$$

where $\mathcal{Z}$ is the z-transform as given in Table 1. This equivalence is shown in Figure 6(b).

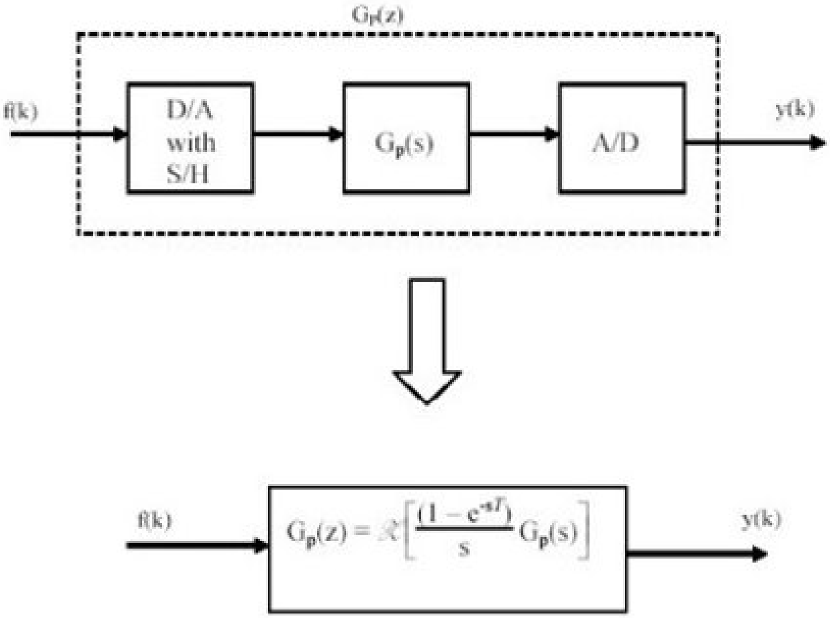


Figure 6: Discretizing a continuous-time plant

As a numerical example, suppose that the continuous-time transfer function of the plant is given by

$$G_{\mathrm{p}}(s) = \frac{4}{s(s+2)}$$

and the sampling interval is $T = 0.2$ seconds. According to equation (2),

$$Y(s) = F^0(s)G_{\mathrm{p}}(s) = \frac{1-e^{-0.2s}}{s}\left[\frac{4}{s(s+2)}\right] = (1-e^{-0.2s})\left[\frac{4}{s^2(s+2)}\right]$$

$$= (1-e^{-0.2s})\left[\frac{-1}{s} + \frac{2}{s^2} + \frac{1}{(s+2)}\right]$$

Using table 1, the discrete-time plant transfer function, for $T = 0.2$, is determined using equation (3) as:

$$G_{\mathrm{p}}(z) = (1-z^{-1})\left[\frac{-z}{z-1} + \frac{2(0.2)z}{(z-1)^2} + \frac{z}{z-e^{-0.4}}\right] = \frac{0.0703z + 0.0616}{(z-1)(z-0.6703)}$$

Knowing $G_{\mathrm{p}}(z)$, and returning to figure 5(b), the control system designer can now proceed to specify the digital controller $G_{\mathrm{c}}(z)$ using classical design techniques to meet the control system requirements. The classical approach to designing a digital controller directly, which has many variations, parallels the classical approach to analog controller design. We begin with simple discrete-time controllers, increasing their complexity until the performance requirements can be met. Classical discrete-time control system design is beyond our scope in this chapter and therefore will not be discussed.

In the following section, we present several methods for discretizing continuous-time controllers. In the latter section, the relationship between continuous-time state variable plant models and their discrete counterparts are derived with the results being useful for designing digital controllers for discrete-time systems.

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Biographical Sketches

Mohammed S. Santina is a recognized expert on advanced control systems with extensive experience in the design and development of satellite control systems for aerospace applications. He received the B.S. and M.S. degrees from California State University, Long Beach (CSULB) and the Ph.D. from the University of California, Irvine (UCI), in 1978, 1981, and 1987, respectively, all in Electrical Engineering. He was with Rockwell International from 1981 to 1992, and with the Aerospace Corporation from1992 to 1996. Since 1996, he has been with the Boeing Company where is currently a Senior Technical Fellow facilitating the integration of attitude control systems technologies for several Boeing programs. He has also taught graduate and undergraduate level courses on control systems and estimation theory for over 22 years at the University of California, Irvine and California State University, Long Beach. He is a co-author of the Digital Control System Design textbook published by Oxford University Press, New York, 1994. He is also the author or co-author of more than forty papers in leading journals including the 5 chapters in the popular Control Handbook, CRC Press, Florida, 1996, and the chapter in the Wiley Encyclopedia of Electrical and Electronics Engineering, John Wiley & Sons, Inc., New York, 1999. His current research interests include time optimal and nonlinear control of agile spacecrafts, and modal estimation and identification of flexible spacecrafts.

Allen R. Stubberud is a Professor of Electrical and Computer Engineering, University of California, Irvine (UCI). He received the B.S. in Electrical Engineering from the University of Idaho in 1956 (with highest honors) and the M.S. and Ph.D. in Engineering from the University of California, Los Angeles (UCLA), in 1958 and 1962, respectively. From 1962 to 1969 he served on the faculty at UCLA as an Assistant/Associate Professor of Electrical Engineering. He has been on the faculty at UCI since 1969. From 1972 to 1996 he has held a number of administrative positions at UCI including the Dean of Engineering, Associate Dean of Engineering and Chairman of the Electrical and Computer Engineering Department. From 1987 to 1988, Professor Stubberud served as the Division Director of the National Science Foundation. He also held the position of Chief Scientist, US Air Force from 1983 to 1985. Professor Stubberud has consulted for a number of organizations, including Jet Propulsion Laboratory, McDonnell-Douglas Corp., ORINCON Corporation, US Air Force, Interstate Electronics and Point Research Corporation. He is author or co-author of more than 200 articles and several textbooks on estimation and control.

Dr. Stubberud has received many honors and awards in recognition of his contributions to the controls and signal processing fields. He is a Fellow of the Institute of Electrical and Electronics Engineers (IEEE), American Institute of Aeronautics and Astronautics (AIAA), American Association for the Advancement of Science (AAAS), and New York Academy of Sciences (NYAS). Dr. Stubberud is the Recipient of the IEEE Centennial medal, IEEE Millennium Medal, and the US Air Force Exceptional Civilian Service Decoration (2 times). He is a member of the North Atlantic Treaty Organization, RTO Mission Systems Panel, and a member of the International Symposium on Circuits and Systems Advisory Committee.