THE LAGRANGIAN SOLUTIONS

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Summary

This chapter focuses on the dynamics in a neighborhood of the five equilibrium points of the Restricted Three-Body Problem. The first section is devoted to the discussion of the linear behavior near the five points. Then, the motion in the vicinity of the collinear points is considered, discussing the effective computation of the center manifold as a tool to describe the nonlinear dynamics in an extended neighborhood of these points. This technique is then applied to the Earth-Moon case, showing the existence of periodic and quasi-periodic motions, including the well-known Halo orbits.

Next, the dynamics near the triangular points is discussed, showing how normal forms can be used to effectively describe the motion nearby. The Lyapunov stability is also considered, showing how the stability is proved in the planar case, and why it is not proved in the spatial case. This section also discusses how to bound the amount of diffusion that could be present in the spatial case. Finally, in the last section we focus on the effect of perturbations. More concretely, we mention the Elliptic Restricted Three-Body Problem, the Bicircular problem and similar models that contain periodic and quasi-periodic time-dependent perturbations.

1. Introduction

Let us consider two point masses (usually called primaries) that attract each other according to the gravitational Newton’s law. Let us assume that they are moving in circular orbits around their common center of mass, and let us consider the motion of an infinitesimal particle (here, infinitesimal means that its mass is so small that we neglect the effect it has on the motion of the primaries and we only take into account the effect
of the primaries on the particle) under the attraction of the two primaries. The study of the motion of the infinitesimal particle is what is known as the Restricted Three-Body Problem, or RTBP for short.

To simplify the equations of motion, let us take units of mass, length and time such that the sum of masses of the primaries, the gravitational constant and the period of the motion of the primaries are 1, 1 and \(2\pi\) respectively. With these units the distance between the primaries is also equal to 1. We denote as \(\mu\) the mass of the smaller primary (the mass of the bigger is then \(1-\mu\)), \(\mu \in (0, \frac{1}{2}]\).

The usual system of reference is defined as follows: the origin is taken at the center of mass of the primaries, the \(X\) -axis points to the bigger primary, the \(Z\) -axis is perpendicular to the plane of motion, pointing in the same direction as the vector of angular momentum of the primaries with respect to their common center of mass, and the \(Y\) -axis is defined such that we obtain an orthogonal, positive-oriented system of reference. With this we have defined a rotating system of reference, that is usually called synodic. In this system, the primary of mass \(\mu\) is located at the point \((\mu -1,0,0)\) and the one of mass \(1-\mu\) is located at \((\mu,0,0)\), see Figure 1.

![Figure 1. The five equilibrium points of the RTBP. The graphic corresponds to the Earth-Moon case. The unit of distance is the Earth-Moon distance, and the unit of mass is the total mass of the system. In these units, the mass of the Moon is \(\mu \approx 0.01215\).](image)

Defining momenta as \(P_X = \dot{X} - Y\), \(P_Y = \dot{Y} + X\) and \(P_Z = \dot{Z}\), the equations of motion can be written in Hamiltonian form. The corresponding Hamiltonian function is

\[
H = \frac{1}{2} \left( P_x^2 + P_y^2 + P_z^2 \right) + YP_x - XP_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},
\]

(1)
being \( r_1^2 = (X - \mu)^2 + Y^2 + Z^2 \) and \( r_2^2 = (X - \mu + 1)^2 + Y^2 + Z^2 \) (see, for instance, Szebehely (1967) for details).

It is well-known that the system defined by (1) has five equilibrium points. Two of them can be found as the third vertex of the two equilateral triangles that can be formed using the two primaries as vertices (usually called \( L_{4,5} \) or Lagrangian points). The other three lie on the \( X \)-axis and are usually called \( L_{1,2,3} \) or Eulerian points (see Figure 1). A more detailed discussion on the existence of these points can be found in many textbooks, like Szebehely (1967). Note that “our” \( L_1 \) and \( L_2 \) are swapped with respect to that reference. This lack of agreement for the definition of \( L_{1,2} \) is rather common in the literature: usually, books on celestial mechanics use the same notation as in Szebehely (1967) but books on astrodynamics follow the convention we use here.

In this chapter we will focus on the dynamics around these points, especially for examples from the Solar system. We will also comment on the main perturbations that appear in astronomical and astronautical applications and their effects.

2. Linear Behavior

In this section we will first discuss the linearization of the dynamics around the five equilibrium points. The presentation is done in a way that prepares the following sections.

2.1. The Collinear Points

Let us define, for \( j = 1, 2 \), \( \gamma_j \) as the distance from the smaller primary (the one of mass \( \mu \) ) to the point \( L_j \), and \( \gamma_3 \) as the distance from the bigger primary to \( L_3 \). It is well-known (see, for instance, Szebehely (1967) that \( \gamma_j \) is the only positive solution of the Euler quintic equation,

\[
\gamma^5_j + (3 - \mu)\gamma^4_j + (3 - 2\mu)\gamma^3_j - \mu\gamma^2_j \pm 2\mu\gamma_j - \mu = 0, \quad j = 1, 2,
\]

\[
\gamma^5_3 + (2 + \mu)\gamma^4_3 + (1 + 2\mu)\gamma^3_3 - (1 - \mu)\gamma^2_3 - 2(1 - \mu)\gamma_3 - (1 - \mu) = 0,
\]

where the upper sign in the first equation is for \( L_1 \) and the lower one for \( L_2 \). These equations can be solved numerically by the Newton method, using the starting point \((\mu/3)^{1/3}\) for the first equation (\( L_{4,2} \) cases), and \(1 - \frac{2}{15}\mu\) for the second one (\( L_3 \) case).

The next step is to translate the origin to the selected point \( L_j \). Moreover, since in Section 3 we will need the power expansion of the Hamiltonian at these points, we therefore perform a suitable scaling in order to avoid fast growing (or decreasing) coefficients. The idea is to have the closest singularity (the body of mass \( \mu \) for \( L_{4,2} \) or the one of mass \( 1 - \mu \) for \( L_3 \)) at distance 1 (see Richardson, 1980b). As the scalings are
not symplectic transformations, let us consider the following process: first we write the differential equations related to (1) and then, on these equations, we perform the following substitution

\[
X = \mp \gamma_j x + \mu + \alpha_j, \\
Y = \mp \gamma_j y, \\
Z = \gamma_j z,
\]

where the upper sign corresponds to \( L_{1,2} \), the lower one to \( L_3 \) and \( \alpha_1 = -1 - \gamma_1 \), \( \alpha_2 = -1 - \gamma_2 \) and \( \alpha_3 = \gamma_3 \). Note that the unit of distance is now the distance from the equilibrium point to the closest primary.

In order to expand the nonlinear terms, we will use that

\[
\frac{1}{\sqrt{(x-A)^2 + (y-B)^2 + (z-C)^2}} = \frac{1}{D} \sum_{n=0}^{\infty} \left( \frac{\rho}{D} \right)^n \left( P_n \left( \frac{Ax + By + Cz}{D\rho} \right) \right),
\]

where \( A, B, C, D \), are real numbers with \( D^2 = A^2 + B^2 + C^2 \), \( \rho^2 = x^2 + y^2 + z^2 \) and \( P_n \) is the polynomial of Legendre of degree \( n \). After some calculations, one obtains that the equations of motion can be written as

\[
\begin{align*}
\dot{x} - 2y - (1 + 2c_2) x &= \frac{\partial}{\partial x} \sum_{n>3} c_n(\mu) \rho^n P_n \left( \frac{x}{\rho} \right), \\
\dot{y} + 2x + (c_2 - 1) y &= \frac{\partial}{\partial y} \sum_{n>3} c_n(\mu) \rho^n P_n \left( \frac{x}{\rho} \right), \\
\dot{z} + c_2 z &= \frac{\partial}{\partial z} \sum_{n>3} c_n(\mu) \rho^n P_n \left( \frac{x}{\rho} \right),
\end{align*}
\]

where the left-hand side contains the linear terms and the right-hand side contains the nonlinear ones. The coefficients \( c_n(\mu) \) are given by

\[
\begin{align*}
c_n(\mu) &= \frac{1}{\gamma_j} \left( (1-\mu)^{n+1} \gamma_j \right), \quad \text{for } L_j, \ j = 1, 2 \\
c_n(\mu) &= \left( -1 \right)^n \frac{\mu^{n+1}}{\gamma_3} \left( 1 + \mu \gamma_3 \right), \quad \text{for } L_3.
\end{align*}
\]

In the first equation, the upper signs are for \( L_1 \) and the lower one for \( L_2 \). Note that these equations can be written in Hamiltonian form, by defining the momenta \( p = \dot{x} - y \).
\( p_y = y + x \) and \( p_z = z \). The corresponding Hamiltonian is then given by

\[
H = \frac{1}{2} \left( p_x^2 + p_y^2 + p_z^2 \right) + yp_x - xp_y - \sum_{n \neq 2} c_n (\mu) p^n P_n \left( \frac{x}{\rho} \right).
\] (3)

The nonlinear terms of this Hamiltonian can be expanded by means of the well-known recurrence of the Legendre polynomials \( P_n \). For instance, if we define

\[
T_n(x, y, z) = p^n P_n \left( \frac{x}{\rho} \right),
\] (4)

then, it is not difficult to check that \( T_n \) is a homogeneous polynomial of degree \( n \) that satisfies the recurrence

\[
T_n = \frac{2n-1}{n} x T_{n-1} - \frac{n-1}{n} \left( x^2 + y^2 + z^2 \right) T_{n-2},
\] (5)

starting with \( T_0 = 1 \) and \( T_1 = x \).

The linearization around the equilibrium point is given by the second order terms of the Hamiltonian (linear terms must vanish) that, after some rearranging, take the form,

\[
H_2 = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + yp_x - xp_y - c_2 x^2 + \frac{c_2}{2} y^2 + \frac{1}{2} p_z^2 + \frac{c_2}{2} z^2.
\] (6)

It is not difficult to derive intervals for the values of \( c_2 = c_2(\mu) \) when \( \mu \in [0, \frac{1}{2}] \) (see Figure 2). As \( c_2 > 0 \) (for the three collinear points), the vertical direction is an harmonic oscillator with frequency \( \omega_2 = \sqrt{c_2} \). Now let us focus on the planar directions, i.e.,

\[
H_2 = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + yp_x - xp_y - c_2 x^2 + \frac{c_2}{2} y^2,
\] (7)

where, for simplicity, we keep the name \( H_2 \) for the Hamiltonian.

Now, let us define the matrix \( M \) as \( J \) Hess(\( H_2 \)).

\[
M = \begin{bmatrix}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
2c_2 & 0 & 0 & 1 \\
0 & -c_2 & -1 & 0
\end{bmatrix}.
\] (8)
The characteristic polynomial is  \( p(\lambda) = \lambda^4 + (2 - c_2)\lambda^2 + \left(1 + c_2 - 2c_2^2\right) \). Calling  \( \eta = \lambda^2 \), we have that the roots of  \( p(\lambda) = 0 \) are given by

\[
\eta_1 = \frac{c_2 - 2 - \sqrt{9c_2^2 - 8c_2}}{2}, \quad \eta_2 = \frac{c_2 - 2 + \sqrt{9c_2^2 - 8c_2}}{2}.
\]

As  \( \mu \in [0, \frac{1}{2}] \) we have that  \( c_2 > 1 \) that forces  \( \eta_1 < 0 \) and  \( \eta_2 > 0 \). This shows that the equilibrium point is a center×center×saddle. Thus, let us define  \( \omega_i \) as  \( \sqrt{-\eta_i} \) and  \( \lambda_i \) as  \( \sqrt{\eta_i} \). For the moment, we do not specify the sign taken for each value (this will be discussed later on).

![Figure 2. Values of \( c_2(\mu) \) (vertical axis), for \( \mu \in [0, \frac{1}{2}] \) (horizontal axis), for the cases \( L_{1,2,3} \).](image)

Now, we want to find a symplectic linear change of variables casting (7) into its real normal form (by real we mean with real coefficients) and, hence, we will look for the eigenvectors of matrix (8). As usual, we will take advantage of the special form of this matrix: if we denote by  \( M_\lambda \) the matrix  \( M - \lambda I_4 \), then

\[
M_\lambda = \begin{bmatrix} A_\lambda & I_2 \\ B & A_\lambda \end{bmatrix}, \quad A_\lambda = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}, \quad B = \begin{bmatrix} 2c_2 & 0 \\ 0 & -c_2 \end{bmatrix}.
\]

Now, the kernel of  \( M_\lambda \) can be found as follows: denoting as  \( \begin{bmatrix} w_1^T & w_2^T \end{bmatrix}^T \) the elements of the kernel, we start solving  \( (B - A^2)w_1 = 0 \) and then  \( w_2 = -Aw_1 \). Thus, the
eigenvectors of $\mathbf{M}$ are given by $\left[2\lambda, \quad \lambda^2 - 2c_2 - 1, \quad \lambda^2 + 2c_2 + 1, \quad \lambda^3 + (1 - 2c_2)\lambda\right]^T$, where $\lambda$ denotes the eigenvalue.

Let us consider now the eigenvectors related to $\omega_i$. From $p(\lambda) = 0$, we obtain that $\omega_i$ verifies

$$\omega_i^4 - (2 - c_2)\omega_i^2 + \left(1 + c_2 - 2c_2^2\right) = 0.$$ 

We also apply $\lambda = \sqrt{-1}\omega_i$ to the expression of the eigenvector and, separating real and imaginary parts as $\mathbf{u}_{\omega_i} + \sqrt{-1}\mathbf{v}_{\omega_i}$ we obtain

$$\mathbf{u}_{\omega_i} = \left(0, -\omega_i^2 - 2c_2 - 1, -\omega_i^2 + 2c_2 + 1, 0\right)^T,$$

$$\mathbf{v}_{\omega_i} = \left(2\omega_i, 0, 0, -\omega_i^3 + (1 - 2c_2)\omega_i\right)^T.$$ 

Now, let us consider the eigenvalues related to $\pm\omega_i$,

$$\mathbf{u}_{\pm\omega_i} = \left(2\lambda, \lambda^2 - 2c_2 - 1, \lambda^2 + 2c_2 + 1, \lambda^3 + (1 - 2c_2)\lambda\right)^T,$$

$$\mathbf{v}_{\pm\omega_i} = \left(-2\lambda, \lambda^2 - 2c_2 - 1, \lambda^2 + 2c_2 + 1, -\lambda^3 - (1 - 2c_2)\lambda\right)^T.$$ 

We consider, initially, the change of variables $\mathbf{C} = \left(\mathbf{u}_{\pm\omega_i}, \mathbf{u}_{\omega_i}, \mathbf{v}_{\pm\omega_i}, \mathbf{v}_{\omega_i}\right)$. To know whether this matrix is symplectic or not, we check $\mathbf{C}^T\mathbf{JC} = \mathbf{J}$. It is a tedious computation to see that

$$\mathbf{C}^T\mathbf{JC} = \begin{bmatrix} 0 & \mathbf{D} \\ -\mathbf{D} & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_{\omega_i} & 0 \\ 0 & d_{\omega_i} \end{bmatrix}.$$ 

This implies that we need to apply some scaling on the columns of $\mathbf{C}$ in order to have a symplectic change. The scaling is given by the factors

$$d_{\omega_i} = 2\lambda_i \left((4 + 3c_2)\lambda_i^2 + 4 + 5c_2 - 6c_2^2\right), \quad d_{\omega_i} = \omega_i \left((4 + 3c_2)\omega_i^2 - 4 - 5c_2 + 6c_2^2\right).$$ 

Thus, we define $s_1 = \sqrt{d_{\omega_i}}$ and $s_2 = \sqrt{d_{\omega_i}}$. As we want the change to be real, we have to require $d_{\omega_i} > 0$ and $d_{\omega_i} > 0$. It is not difficult to check that this condition is satisfied for $0 < \mu < \frac{1}{2}$ in all the points $L_{1,2,3}$ if $\lambda_i > 0$ and $\omega_i > 0$.

To obtain the final change, we have to take into account the vertical direction $(z, p_z)$: to put it into real normal form we use the substitution
This implies that the final change is given by the symplectic matrix

\[
C = \begin{bmatrix}
\frac{2\lambda}{\eta_1} & 0 & 0 & -\frac{2\lambda}{\eta_1} & \frac{2\eta_1}{\eta_2} & 0 \\
\frac{\lambda^2-2c_1-1}{\eta_1} & -\eta_1^2-2c_1-1 & 0 & \frac{\lambda^2-2c_1-1}{\eta_1} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{\eta_0}} & 0 & 0 & 0 \\
\frac{\lambda^2+2c_1+1}{\eta_1} & -\eta_1^2+2c_1+1 & 0 & \frac{\lambda^2+2c_1+1}{\eta_1} & 0 & 0 \\
\frac{\lambda^2-(1-2c_1)\lambda}{\eta_1} & 0 & 0 & -\frac{\lambda^2-(1-2c_1)\lambda}{\eta_1} & -\eta_1^2+(1-2c_1)\eta_0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\eta_2}
\end{bmatrix}
\]  

that casts Hamiltonian (6) into its real normal form,

\[
H_2 = \lambda_1 x p_x + \frac{\omega_1}{2} y^2 + p_y^2 + \frac{\omega_2}{2} (z^2 + p_z^2)
\]  

where, for simplicity, we have kept the same name for the variables. Later on we will use a complex normal form for \( H_1 \) because it will simplify the computations. This complexification is given by

\[
x = q_1, \quad y = \frac{q_2 + \sqrt{-1} p_2}{\sqrt{2}}, \quad z = \frac{q_3 + \sqrt{-1} p_3}{\sqrt{2}},
\]

\[
p_x = p_1, \quad p_y = \frac{\sqrt{-1} q_2 + p_2}{\sqrt{2}}, \quad p_z = \frac{\sqrt{-1} q_3 + p_3}{\sqrt{2}},
\]

and it puts (10) into its complex normal form,

\[
H_2 = \lambda_1 q_1 p_1 + \sqrt{-1} \omega_1 q_2 p_2 + \sqrt{-1} \omega_2 q_3 p_3,
\]

being \( \lambda_1, \omega_1 \) and \( \omega_2 \) real (and positive) numbers.
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Biographical Sketch

Àngel Jorba (born in 1963 in Barcelona, Spain) received his PhD from the University of Barcelona in 1991 under the supervision of Carles Simó. He has been associate professor at the Polytechnic University of Catalonia and is currently Professor of Applied Mathematics at the University of Barcelona. He is a member of the editorial board of Discrete and Continuous Dynamical Systems - Series B since 2001, and he has served as coordinator of the Spanish network of dynamical systems (DANCE) from 2006 to 2010. His research interests include celestial mechanics and astrodynamics, with a particular interest in the analysis of space missions. He is also interested on the occurrence of quasi-periodic motions in dynamical systems, and in the development of numerical and semi-analytical tools to deal with the application of dynamical systems to real situations.

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