CONTROLLABILITY, OBSERVABILITY, AND STABILITY OF MATHEMATICAL MODELS

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Contents

Summary

This chapter presents an overview of three fundamental concepts in Mathematical System Theory: controllability, stability and observability. These properties play a prominent role in the study of mathematical models and in the understanding of their behavior. They constitute the main research subject in Control Theory. Historically the

tools and techniques of Automatic Control have been developed for artificial engineering systems but nowadays they are more and more applied to “natural systems”. The main objective of this chapter is to show how these tools can be helpful to model and to control a wide variety of natural systems.

1. Introduction

The main goal of this chapter is to develop in some details some notions of Control Theory introduced in *Basic principles of mathematical modeling*. It concerns more specifically three structural properties of control systems: controllability, stability and observability. Based on the references listed in the end of this chapter, we give a survey of these three properties with applications to various LSS examples.

By a control system we mean dynamical system evolving in some state space and that can be controlled by the user. More precisely we are interested in the study of systems that can be modeled by differential (respectively difference) equations of the form

$$\begin{cases} \dot{x}(t) = \dfrac{dx}{dt} = X\left(x(t), u(t)\right), \\ y(t) = h\left(x(t)\right), \end{cases} \tag{1}$$

where the variable t represents the time, the vector $x(t)$ is the state of the system at time t, the vector

$u(t)$ is the input or the control, i.e., the action of the user or of the environment and the vector $y(t)$ is the output of the system that is, the available information that can be measured or observed by the user. The dynamics function X indicates how the system changes over time.

We shall address the following problems:

- When the whole state $x(t)$ is not available for measurement, how is it possible to use the information provided by $y(t)$ together with the dynamics (1) in order to get a “good” estimation of the real state of the system? This turns out to be an observability problem.
- How to use the control $u(t)$ in order to meet some specified needs? This is the problem of controllability and stabilizability.

To illustrate these various concepts, we explain them through a simple LSS system: an epidemic model for the transmission of an infectious disease. A homogeneous population is divided into four classes S, E, I and T according to the health of its individuals.

Let $S(t)$ denote the number of individuals who are susceptible to the disease, i.e., who are not yet infected at time t. $E(t)$ denotes the number of members at time t who are exposed but not yet infected. $I(t)$ denotes the number of infected individuals, that is, who are infectious and able to spread the disease by contact with individuals who are susceptible. $T(t)$ is the number at time t of treated individuals. The total population size is denoted $N = S + E + I + T$. The dynamic of the disease can be described by the following differential system:

$$\begin{cases} \dfrac{dS}{dt} = bN - \mu S - \beta \dfrac{SI}{N}, \\ \dfrac{dE}{dt} = \beta \dfrac{SI}{N} - (\mu + \varepsilon) E, \\ \dfrac{dI}{dt} = \varepsilon E - (r + d + \mu) I, \\ \dfrac{dT}{dt} = rI - \mu T, \\ \dfrac{dN}{dt} = (b - \mu) N - dI, \end{cases} \tag{2}$$

where the parameter b is the rate for natural birth and μ that of natural death. The parameter β is the transmission rate, d is the rate for disease-related death, ε is the rate at which the exposed individuals become infective and r is the per-capita treatment rate. The parameters b, μ, β, d and ε are assumed to be constant. The per-capita treatment rate may vary with time.

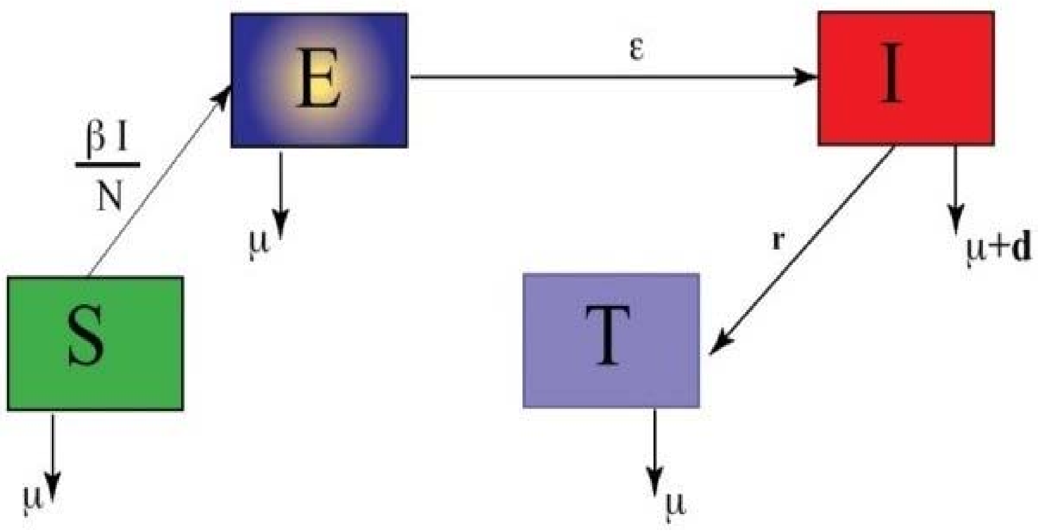

Fig 1: This diagram illustrates the dynamical transfer of the population.

The system (2) can be written by using the fractions $s = S/N$, $e = E/N$, $i = I/n$ and $\tau = T/N$ of the classes S, E, I and T in the population. These fractions satisfy the system of differential equations:

$$\begin{cases} \dfrac{ds}{dt} = b - bs - (\beta - d)si, \\ \dfrac{de}{dt} = \beta si - (b+\varepsilon)e + dei, \\ \dfrac{di}{dt} = \varepsilon e - (r+d+b)i + di^2, \\ \dfrac{d\tau}{dt} = ri - b\tau + di\tau. \end{cases} \tag{3}$$

Since $s + e + i + \tau = 1$, it is sufficient to consider the following system

$$\begin{cases} \dfrac{ds}{dt} = b - bs - (\beta - d)si, \\ \dfrac{de}{dt} = \beta si - (b+\varepsilon)e + dei, \\ \dfrac{di}{dt} = \varepsilon e - (r+d+b)i + di^2. \end{cases} \tag{4}$$

If we suppose that the proportion of infected individuals can be measured and that we can control the population by choosing the treatment rate then the above system can be seen as a control system of the form (1) with: the state of the system is $x(t) = (s(t), e(t), i(t))$, the control is $u(t) = r(t)$ and the measurable output is $y(t) = i(t)$. The state space is

$$\Omega = \left\{ x \in \mathbb{R}^3 : 0 \le s \le 1,\ 0 \le e \le 1,\ 0 \le i \le 1,\ s + e + i \le 1 \right\}.$$

The aim of Control Theory is to give answers to the following questions:

1. Given two configurations x_1 and x_2 of the system (3), is it possible to steer the system from x_1 to x_2 by choosing an appropriate treatment strategy r(t)? For a given initial state $x_0 = (s_0, e_0, i_0)$ and a given time T, what are all the possible states that can be reached in time T by using all the possible control functions r(t)?
2. Does the system (3) possess equilibrium points? Are they stable or unstable? How can the control be chosen in order to make a given equilibrium (for instance, the disease-free equilibrium) globally asymptotically stable?
3. Since only some variables can be measured (for instance, the proportion i(t) of infected individuals in the above epidemic example), is it possible to use them together with the dynamics (3) of the system in order to estimate the non measurable variables s(t) and e(t) and how can this be done?

Questions 1 involve the controllability of control systems. We shall give in Section 2 a general definition of this notion as well as some simple useful criteria that allow us to study the controllability of some nonlinear systems. Questions 2 invoke the stability and the stabilization of nonlinear system, this will be the subject of Section 3 where some classical stability results are exposed. Questions 3 are connected to the observability and the construction of observers for dynamical systems. These problems will be addressed in Section 4 and Section 5.

In each section we shall apply the different tools to various models such as predator-prey systems, fisheries and bioreactors. Concerning the epidemic model (2-4), the divers problems mentioned above are still under investigation. We can give here just some partial answers. It can be shown that if the per-capita treatment rate satisfies

$$r \geq r_0 = \frac{\beta\varepsilon}{\varepsilon + b} - d - b \tag{5}$$

then the system has a unique equilibrium point in the domain Ω. This equilibrium is the disease free equilibrium $s = 1,\ e = i = \tau = 0$ and it is globally asymptotically stable, that is, if $\left(s(0),\ e(0),\ i(0)\right) \in \Omega$ is any initial condition then the solution of the system (4) starting from this state will converge to the disease free equilibrium, i.e., $s(t) \to 1,\ e(t) \to 0$ and $i(t) \to 0$ as $t \to \infty$. Therefore, if the treatment satisfies the condition (5) then the disease will die out. If $r < r_0$ then the disease free equilibrium becomes unstable and in this case, there is another equilibrium which belongs to the interior of the domain Ω. This equilibrium is the endemic equilibrium and it is globally asymptotically stable within the interior of Ω provided that $r < r_0$. The disease will be endemic in this case.

2. Controllability

In this section, we present an elementary overview of an important property of a system, namely that of controllability. This concept has been briefly introduced and explained in *Basic principles of mathematical modeling*. Here, we deal with the problem of testing controllability of systems of the type

$$\dot{x}(t) = X\left(x(t), u(t)\right). \tag{6}$$

The state vector $x(t)$ belongs to the state space M which will always be here $\mathbb{R}^n$ or an open connected subset of $\mathbb{R}^n$. The control function $t \mapsto u(t)$ are defined on $[0, \infty)$ and take values in a connected subset U of $\mathbb{R}^m$. These control functions are assumed to belong to an *admissible* control set $\mathcal{U}_{ad}$. This admissible control set is generally specified by the problem considered. Here it is assumed that $\mathcal{U}_{ad}$ contains the set of piecewise constant functions as a dense subset, i.e., any admissible control function can be "approximated" by piecewise constant functions. The dynamics function $X : M \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be analytic on (x, u). For each control value $u \in U$, we denote by X^u the vector field ("vector function") defined by $X^u(x) = X(x,u)$ for all $x \in M$.

A real-valued function f defined on some open set $\mathcal{O} \subset \mathbb{R}$ is said to be analytic if for each $x_0 \in \mathcal{O}$, there exists a positive real number $r > 0$ such that f(x) is the sum of a power series for all x with $|x - x_0| < r$:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \textit{for all } x \textit{ satisfying} |x - x_0| < r.$$

For an analytic function, the coefficients a_n can be computed as $a_n = \frac{f^{(n)}(a)}{n!}$

In a similar way a function $f : \mathcal{V} \subset \mathbb{R}^k \to \mathbb{R}$ of several variables is said to be analytic if it is locally given by power series.

A vector function

$$X: \quad \mathcal{V} \subset \mathbb{R}^k \quad \to \mathbb{R}^n$$

$$x = (x_1, \ldots, x_k) \mapsto X(x) = \begin{pmatrix} X_1(x_1, \ldots, x_k) \\ \vdots \\ X_n(x_1, \ldots, x_k) \end{pmatrix}$$

is analytic if all its components X_i are analytic.

For each $x \in M$ and each control function $u(.) \in \mathcal{U}_{ad}$, we denote by $X_t^{u(.)}(x)$ the solution of (6) satisfying $X_0^{u(.)}(x) = x$. We assume moreover that $X_t^{u(.)}(x)$ is defined for all $t \in [0, \infty)$. For instance, if X is a linear vector function, that is, $X(x,u) = Ax + Bu$, with A and B being matrices with appropriate dimensions, then $X_t^{u(.)}(x) = e^{tA}x + \int_0^t e^{(t-s)A} Bu(s)\, ds$.

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Biographical Sketch

Abderrahman Iggidr, was born in Rabat (Morocco). He received the D.E.A in 1989 in applied mathematics and the Ph.D. degree in 1992, in mathematics and automatic control, from the University of Metz (France). Since 1993, he has been researcher at INRIA (The French National Institute for Research in Computer Science and Control). His research interests are the field of nonlinear systems, controllability, observability, stabilization, observers, and stabilization by estimated state feedback from observers. His research interests also include the application of control theory to the modeling and the regulation of biological systems.