# METHODS OF INTEGRAL TRANSFORMS

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## Contents

**Summary**

The methods of integral transforms are very efficient to solve and research differential and integral equations of mathematical physics. These methods consist in the integration of an equation with some weight function of two arguments that often results in the simplification of a given initial problem. The main condition for the application of an integral transform is the validity of the inversion theorem which allows one to find an unknown function knowing its image. The Fourier, Laplace, Mellin, Hankel, Meyer, Hilbert, and other transforms are used depending on a weight function and an integration domain. With the help of these transforms many problems of the oscillation theory, heat conductivity, neutron diffusion and slowing-down, hydrodynamics, the elasticity theory, and physical kinetics can be solved.

## 1. Introduction

*An integral transform is a functional transform of the form*

$$F(x) = \int_{\Gamma} K(x,t) f(t) dt \,,$$

where $\Gamma$ is a finite or infinite contour in the complex plane, $K(x,t)$ is the kernel of a transform. The most frequently considered integral transforms are those for which $K(x,t) = K(xt)$ and $\Gamma$ is the real axis or its part $(a,b)$. If $-\infty < a,b < \infty$, then a transform is referred to as a *finite integral transform.* Formulae that allow us to reconstruct a function $f(t)$ by a given one $F(x)$ are said to be *inversion formulae* of integral transforms.

If $x,t \in \mathbf{R}^n$ and $\Gamma$ is a domain in the *n*-dimensional Euclidean space, then *multiple (multidimensional) integral transforms* are considered.

Integral transforms are frequently applied when solving differential and integral equations and their choice depends on the type of a considered equation. The main condition when choosing an integral transform is the possibility to reduce a differential or integral expression to a more simple differential equation (or, what is better, to an algebraic ratio) with respect to a function $F(x)$ implying that an inversion formula is known. If a contour $\Gamma$ is finite (for example, a segment), then a transform $F(x)$ is called a finite transform of $f(t)$. It is obvious that the number of integral transforms can be considerably increased by introducing new kernels.

In the following we mainly consider transforms where a contour $\Gamma$ is the real axis or real semiaxis supposing that all integrals are finite.

## 2. Basic Integral Transforms

### 2.1. The Fourier Transform

The expression

$$F(x) \equiv \mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\tau} f(\tau) d\tau$$

is called the *Fourier transform* of a function $f(t)$.

A function $F(x)$ is called the *Fourier image* of a function $f$ . The *inverse Fourier transform* has the form

$$f(t) \equiv \mathcal{F}^{-1}[F(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} F(x) dx .$$

Combining these expressions, we come to the *exponential Fourier formula*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\tau} \int_{-\infty}^{\infty} e^{-ix\tau} f(\tau) d\tau dx , \tag{1}$$

which is equivalent to the *integral Fourier formula*

$$f(t) = \frac{1}{\pi} \int_{0}^{\infty} dx \int_{-\infty}^{\infty} f(\tau) \cos(x(\tau - t)) d\tau . \tag{2}$$

Rearranging the cosine of difference, we obtain the identity

$$f(t) = \int_{0}^{\infty} [a(x)\cos(tx) + b(x)\sin(tx)] dx , \tag{3}$$

where

$$a(x)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(t)\cos xt\,dt,\quad b(x)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(t)\sin xt\ \ dt\,.$$

If $f(t)$ is an even function, then the formula (3) takes the form

$$f(t)=\frac{2}{\pi}\int\limits_{0}^{\infty}\cos\, tx\ du\int\limits_{0}^{\infty} f(\tau)\cos x\tau\ d\tau\,. \tag{4}$$

Similarly, if $f(t)$ is an odd function, then

$$f(t)=\frac{2}{\pi}\int\limits_{0}^{\infty}\sin tx\ dx\int\limits_{0}^{\infty} f(\tau)\sin x\tau\ d\tau. \tag{5}$$

Conditions on a function $f$, whereby the formulae (1), (2) are valid, and the direct and inverse Fourier transforms are determined in the following theorem, where $L(-\infty,+\infty)$ denotes a space of functions intergrable in the sense of Lebesgue over $(-\infty,+\infty)$.

**Theorem 1.** *Assume that* $f\in L(-\infty,+\infty)$ *is a function with a bounded variation on any finite interval. Then the formulae* (1), (2) *are valid if we replace their left-hand sides by* $[f(t+0)+f(t-0)]/2$ *at the points of discontinuity of* $f(t)$.

#### 2.1.1. Basic Properties of the Fourier Transform

If a function $f(t)$ is integrable over the interval $(-\infty,\infty)$, then a function $F(x)$ exists for all $t$. The functions $F(x)$ and $f(t)$, the former being the Fourier transform of the latter, are together called a *couple of the Fourier transforms* (*or a Fourier transform pair*). Assume that

$$F_c(x)=\sqrt{\frac{2}{\pi}}\int\limits_{0}^{\infty} f(t)\cos\ xt\ dt\,, \tag{6}$$

then from the formula (4) it follows that

$$f(t)=\sqrt{\frac{2}{\pi}}\int\limits_{0}^{\infty} F_c(x)\cos\ tx\ dx\,. \tag{7}$$

The functions connected in such a way are called a couple of the *Fourier cosine-transforms.* Similarly, from the formula (5) we can obtain a couple of the *Fourier sine-transforms:*

$$F_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin xt \, dt \,, \tag{8}$$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(x) \sin tx \, dx \cdot \tag{9}$$

If $f(t)$ is an even function, then

$$F(x) = F_c(x) ;$$

if $f(t)$ is an odd function, then

$$F(x) = iF_s(x) .$$

Let the functions $F(x)$ and $G(x)$ be the Fourier transforms of functions $f(t)$ and $g(t)$ determined by the formulae (6), (7), respectively.

The functions

$$F(u)G(u) \text{ and } h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau$$

are a couple of the Fourier transforms. The function $h(t)$ is called a convolution of functions $f(t)$ and $g(t)$ and is denoted by as $h = f * g = g * f$ .

**Theorem 2 (about convolution).** *Let* $f, g \in L(-\infty,\infty)$ . *Then* $h(t) = f * g(t)$ *belongs to* $L(-\infty,\infty)$ , *and the function* $\sqrt{2\pi}F(x)G(x)$ *is its Fourier transform. Conversely, the product* $\sqrt{2\pi}F(x)G(x)$ *belongs to* $L(-\infty,\infty)$ , *and its Fourier transform is* $f * g(t)$ .

*The Parseval formulae*. Assume that $f(t) \in L(-\infty,\infty)$ , is intergrable over any finite interval, and

$$G(x) = \frac{1}{\sqrt{2\pi}} \lim_{l\to\infty} \int_{-l}^{l} g(\tau) e^{-x\tau} d\tau$$

for all $x$ , moreover, $G(x)$ is finite everywhere and belongs to $L(-\infty,\infty)$ . Then the *Parseval equality*

$$\int_{-\infty}^{\infty} F(x)G(x)dx = \int_{-\infty}^{\infty} f(t)g(-t)dt \, . \tag{10}$$

holds. In particular, for $f = g$ we have the *Parseval formula*

$$\int_{-\infty}^{\infty} |F(x)|^2 \, dx = \int_{-\infty}^{\infty} |f(t)|^2 \, dt . \tag{11}$$

### 2.1.2. The Multiple Fourier Transforms

By definition we have

$$F(x) \equiv \mathcal{F}[f(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ixt} f(t) dt ,$$

where $x = (x_1, x_2),\ t = (t_1, t_2), xt = x_1 t_1 + x_2 t_2, dt = dt_1 dt_2$ . The function $F(x)$ is called the Fourier transform of a function $f(t)$ of two variables. For functions $f(t)$ and $F(x)$, that belongs to $L(R^2)$, the following inversion formula holds:

$$f(t) \equiv \mathcal{F}^{-1}[F(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixt} F(x) dx .$$

If $x, t \in \mathbf{R}^n$, then

$$F(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-ixt} f(t) dt, \quad f(t) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{ixt} F(x) dx .$$

## 2.2. The Laplace Transform

### 2.2.1. The Laplace Integral

Let $f(t)$ be a function of real variable $t$, $0 \le t < +\infty$ , intergrable in the sense of Lebesgue over any finite interval $(0, A)$. Let $p$ be a complex number. The function

$$F(p) \equiv \mathcal{L}[f(t)] = \int_{0}^{\infty} e^{-pt} f(t) dt \tag{12}$$

is called *the Laplace transform* of a function $f(t)$.

### 2.2.2. The Inversion Formula for the Laplace Transform

Using the definition of the Laplace transform and assuming that $p = \gamma + iy$, from (1), (12) we obtain:

$$\int_{\gamma - iw}^{\gamma + iw} e^{pt} F(p) dp = i e^{\gamma t} \int_{-w}^{+w} e^{ity} dy \int_{0}^{\infty} e^{-\gamma\tau} [e^{-\gamma\tau} f(\tau)] d\tau . \tag{13}$$

But according to (1) for $w \to \infty$ the double integral in the right-hand side of the equation (13) is equal to $2\pi e^{-\gamma t} f(t)$ for $t > 0$ and to zero for $t < 0$. Therefore the equation (13) gives

$$f(t) = \frac{1}{2\pi i} \lim_{w \to \infty} \int_{\gamma - iw}^{\gamma + iw} e^{pt} F(p) dp \qquad (14)$$

for $t > 0$ and zero for $t < 0$. The expression (14) is the *inversion formula* for the Laplace transform. A function $f(t)$ must satisfy conditions providing the existence of the Laplace transform (12) and $\gamma$ must be greater than a real part of any singular point of the Laplace image $F(p)$.

### 2.2.3. Limit Theorems

The following statements (*limit theorems*) are valid

(1) if $F(p)$ is the Laplace transform and $\mathcal{L}(f'(t))$ exists, then

$$\lim_{p \to \infty} pF(p) = f(0+0);$$

(2) if, besides, there exist a limit of $f(t)$ as $t \to \infty$, then

$$\lim_{p \to 0} pF(p) = \lim_{t \to \infty} f(t).$$

## 2.3. The Mellin Transform

*The Mellin transform*

$$F(s) \equiv \mathcal{M}[f(t)] = \int_0^\infty f(t) t^{s-1} dt, \quad s = \sigma + i\tau \qquad (15)$$

is closely connected with the Fourier and Laplace transforms.

The Mellin transform can be successfully applied when solving a certain class of plain harmonic problems in a sectorial domain, problems of the elasticity theory, and also when studying special functions, summing series, and calculating integrals. The theorems concerning the Mellin transform can be obtained from the corresponding theorems for the Fourier and Laplace transforms by the change of variables.

Laplace transforms by the change of variables.

**Theorem 4.** *Let* $t^{\sigma-1} f(t) \in L(0, +\infty)$. *Then the following inversion formula holds:*

$$\frac{f(t+0)+f(t-0)}{2}=\frac{1}{2\pi i}\lim_{\lambda\to\infty}\int_{\sigma-i\lambda}^{\sigma+i\lambda}F(s)t^{-s}ds\,, \tag{16}$$

*where the Mellin image* $F(s)$ *is defined in* (15).

**Theorem 5.** *Let* $F=\mathcal{M}[f], G=\mathcal{M}[g]$ *Let*

*either* $t^{k-1}f(t)\in L(0,+\infty), G(1-k-ix)\in L(-\infty,+\infty)$ ,

*or* $F(k+ix)\in L(-\infty,+\infty), t^k g(t)\in L(0,+\infty)$ . *Then*

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathcal{F}(s)G(1-s)ds=\int_0^{\infty}f(t)g(t)dt\,. \tag{17}$$

*Besides, the following relation holds:*

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}F(s)G(s)ds=\int_0^{\infty}g(t)f\left(\frac{1}{t}\right)\frac{dt}{t}\,. \tag{18}$$

**Theorem 6 (about convolution).** *Assume that* $t^k f(t)$ *and* $t^k g(t)$ *belong to* $L(0,+\infty)$ *and*

$$h(t)=\int_0^{\infty}f(\tau)g\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau}.$$

*Then* $t^k h(t)\in L(0,+\infty)$ *and its Mellin transform is* $F(s)G(s)$ .

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**Biographical Sketches**

**Agoshkov Valery Ivanovich** is a Doctor of Physical and Mathematical Sciences, professor of Institute of Numerical Mathematics of Russian Academy of Sciences (Moscow). He is the expert in the field of computational and applied mathematics, the theory of boundary problems for the partial differential equations and transport equation, the theory of the conjugate operators and their applications. He is also the author of more than 160 research works, including 9 monographs. His basic research works are devoted to:

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