VARIATIONAL FORMULATION OF PROBLEMS AND VARIATIONAL METHODS

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Contents

1. Introduction

1.1. Motivation

Numerous problems from physics can be modeled by partial differential equations. Let us illustrate it by a simple example.

An elastic membrane Ω is a planar bounded domain glued to a rigid curve Γ, but a force $f(x)dx$ presses on each surface element $dx = dx_1 dx_2$. The vertical membrane displacement is represented by a real valued function u, which is the solution of Laplace's equation:

$$-\Delta u(x) = f(x), \qquad x = (x_1, x_2) \in \Omega, \tag{1.1}$$

where the Laplace operator Δ is defined by:

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}.$$

As the membrane is glued to the curve Γ, the function u satisfies the following boundary condition:

$$u(x) = 0, \quad x \in \Gamma. \tag{1.2}$$

This is the homogeneous Dirichlet problem for the Laplace equation.
Our aim here is to show that problem described by Eqs.(1.1)-(1.2) has a unique solution (in a sense to define). In order to answer this question, we use a general approach based on a variational formulation of this boundary value problem.

1.2. Principle of the Method

Let us briefly describe the method, without going into the details, as they form the object of the present chapter. Let us suppose that the problem described by Eqs.(1.1)-(1.2) has a "smooth" solution u (for example, twice differentiable). Let v be any arbitrary function in the space $D(\Omega)$ of indefinitely differentiable functions with compact support in Ω. We multiply Equation (1.1) by $v(x)$ and integrate with respect to x over Ω (f is supposed to be continuous, for example); this gives:

$$\int_\Omega -(\Delta u v)(x)\, dx = \int_\Omega (f v)(x) dx.$$

Let us recall Green's formula (Ω is bounded here)

$$\int_\Omega (\Delta u v)(x)\, dx = -\int_\Omega (\nabla u \cdot \nabla v)(x) dx \ + \ \int_\Gamma (\frac{\partial u}{\partial \nu} v)(x)\, d\Gamma(x), \tag{1.3}$$

which is nothing but the generalization of the integration by parts formula in one dimension. In this formula, $\nabla u(x)$ is the gradient of u at $x = (x_1, x_2)$, i.e., a vector with components $(\partial u/\partial x_1)(x), (\partial u/\partial x_2)(x)$ and $\nabla u(x) \cdot \nabla v(x)$ denotes the inner product in $\mathbb{R}^2$ of vectors $\nabla u(x)$ and $\nabla v(x)$, i.e.,

$$\nabla u(x) \cdot \nabla v(x) = \frac{\partial u}{\partial x_1}(x)\frac{\partial v}{\partial x_1}(x) + \frac{\partial u}{\partial x_2}(x)\frac{\partial v}{\partial x_2}(x);$$

finally, $d\Gamma(x)$ denotes the measure on Γ and $\frac{\partial u}{\partial \nu}(x) = \nabla u(x) \cdot \nu(x)$, where $\nu(x)$ is the unit normal at x of Γ oriented towards the exterior of Ω (Ω is a bounded open set with a Lipschitz-continuous boundary, according to Grisvard's definition [5], so that this normal exists almost everywhere (a.e.) on Γ). Using (1.3) allows us to transform the equation in the following way:

$$\int_\Omega (\nabla u \cdot \nabla v)(x) dx = \int_\Omega (f v)(x) dx,$$

since $v|_\Gamma = 0$. We shall in fact study this new equation, noting that it makes sense for far less regular functions u, v (and also f).

The variational problem is:

find $u \in H$ such that for all $v \in H$, $\mathcal{A}(u,v) = L(v)$, (1.4)

where the bilinear form $\mathcal{A}$ and the linear form L are defined by:

$$\mathcal{A}(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx, \quad L(v) = \int_\Omega f(x) v(x) dx. \qquad (1.5)$$

The functional space H (of Sobolev type) will be explained later.

The strategy is as follows: in order to show that the boundary value problem (1.1)-(1.2) has a solution, we first show its equivalence with the variational problem (1.4)-(1.5), then we show that the variational problem has a unique solution. The precise definition of the space H also belongs to the variational strategy; in order to construct it, we need some of the results contained in the next Section.

2. The Variational Method

2.1. The Functional Framework

We introduce the notation and review some useful results (without proofs) concerning the Sobolev spaces. For an introduction to Sobolev spaces and to the theory of distributions, we refer for example to [1], [3], [8].

Let Ω be an open subset in $\mathbb{R}^n$. We first denote by $L^2(\Omega)$ the space of real valued measurable functions which are square integrable on Ω with respect to the Lebesgue measure. This space is a Hilbert space with the scalar product defined by

$$(u,v)_{L^2(\Omega)} = \int_\Omega u(x) v(x) dx.$$

The associated norm is denoted by $\|u\|_{0,\Omega} = (u,u)^{1/2}_{L^2(\Omega)}$. Given a multi-index for the derivative order $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, we set:

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha n}}, \qquad \text{with } |\alpha| = \alpha_1 + ... + \alpha_n .$$

A distribution on Ω is a linear form T defined on $D(\Omega)$, which is "continuous" in the following sense: for all sequence $(\varphi_n)_{n\in\mathbb{N}}$ converging to φ in $D(\Omega)$, we have: $T(\varphi_n) \to T(\varphi)$, when $n \to +\infty$. We recall that the convergence in $D(\Omega)$ is defined in the following way: $\varphi_n \to \varphi$ in $D(\Omega)$ if there exists a compact set $K, K \subset \Omega$, containing the support of φ and all the supports of the function φ_n, and if, for any $\alpha \in \mathbb{N}^n$, $D^\alpha \varphi_n$ converges uniformly on K to $D^\alpha \varphi$. We denote by $D'(\Omega)$ the set of distributions on Ω and by $< .,. >$ the duality bracket between the spaces $D'(\Omega)$ and $D(\Omega)$. If $T \in D'(\Omega)$, we can define its derivative of any order $\alpha \in \mathbb{N}^n$; it is the distribution, denoted by $D^\alpha T$, defined by: for all $\varphi \in D(\Omega)$, $< D^\alpha T, \varphi > = (-1)^{|\alpha|} < T, D^\alpha \varphi >$.

For any integer $m \ge 1$, the Sobolev space $H^m(\Omega)$ consists of those functions $v \in L^2(\Omega)$ for which the partial derivatives (in the sense of distributions or generalized functions) belong to the space $L^2(\Omega)$, for each multi-index α such that: $|\alpha| \le m$. This space is a Hilbert space with the scalar product defined by:

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} .$$

The associated norm is $\| u \|_{m,\Omega} = (u, u)^{1/2}_{H^m(\Omega)}$. We also use the following semi-norm:

$$|u|_{m,\Omega} = \left(\sum_{|\alpha| = m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} \right)^{1/2} .$$

We denote by $H_0^m(\Omega)$ the closure of $D(\Omega)$ in the space $H^m(\Omega)$. When the set Ω is bounded, we have the following Poincaré-Friedrichs inequality: there exists a positive constant $C_P(\Omega)$, such that:

$$\text{for all } v \in H_0^1(\Omega), \quad |v|_{0,\Omega} \le C_P(\Omega) |v|_{1,\Omega} . \tag{2.1}$$

As an immediate consequence, we get:

Proposition 2.1. *Let* Ω *be bounded. Then the semi-norm* $(v \to |v|_{1,\Omega})$ *is a norm on* $H_0^1(\Omega)$ *which is equivalent to the norm* $(v \to \|v\|_{1,\Omega})$.

More generally, we introduce, for any non negative integer m and any real number p, $1 \le p \le \infty$, the Sobolev space:

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega), D^\alpha v \in L^p(\Omega), \text{ for all } \alpha, |\alpha| \le m\},$$

which is a Banach space with the following norm:

$$\|v\|_{m,p,\Omega} = \left[\sum_{|\alpha| \le m} \int_\Omega |D^\alpha v(x)|^p \, dx\right]^{1/p}, \quad \text{for } p < +\infty$$

or

$$\|v\|_{m,\infty,\Omega} = \max_{|\alpha| \le m} \|D^\alpha v\|_{L^\infty(\Omega),} \quad \text{for } p = +\infty,$$

where

$$\|v\|_{L^\infty(\Omega)} = \text{Inf}\{M \ge 0, \text{ such that } |v| \le M, \text{ a.e. on } \Omega\}.$$

Finally, we need to define the trace on Γ of functions belonging to Sobolev spaces. In the monodimensional case, we can show that a function $v \in H^1(]a,b[)$ is equal (a.e.) to a continuous function on $[a,b]$ (still denoted by v); more precisely there exists a positive constant $C(a,b)$ such that:

$$\sup_{x \in [a,b]} |v(x)| \le C(a,b) \|v\|_{1,\Omega}.$$

In that case, there is no difficulty to define $v(a)$ or $v(b)$. For higher dimensions, we introduce the space $D(\bar{\Omega})$ of restrictions in Ω of functions belonging to $\mathcal{D}(\mathbb{R}^n)$. We suppose from now on that Ω is bounded with at least a Lipschitz-continuous boundary (we refer to [5] for a precise definition of the regularity of the boundary). Then $D(\bar{\Omega})$ is dense in $H^1(\Omega)$, which allows us to define, by density arguments, a trace operator.

Theorem 2.2. *There exists a continuous linear mapping* $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$, *Such that for all* $v \in D(\bar{\Omega})$, $\gamma_0(v) = v|_\Gamma$. *The kernel of this mapping is the space* $H_0^1(\Omega)$ *and its image, denoted by* $H^{1/2}(\Gamma)$, *is dense in* $L^2(\Gamma)$. *Conversely, any function* g *in* $H^{1/2}(\Gamma)$ *can be extended to a function* v *in* $H^1(\Omega)$, *but this extension is not unique (if* v *is an extension, the other ones are of the form* $v + w$, *where* w *is an arbitrary function in* $H_0^1(\Omega)$*). The space* $H^{1/2}(\Gamma)$ *is a Hilbert space with the norm defined by:*

$$\|g\|_{H^{1/2}(\Gamma)} = \inf_{v \in H^1(\Omega), \gamma_0(v) = g} \|v\|_{H^1(\Omega)}. \tag{2.2}$$

More generally, let us suppose that Ω has a boundary of class $\mathcal{C}^{1,1}$ (it means in particular that this boundary is locally the graph of a function whose derivatives of order 1 are Lipschitz continuous); we denote by ν the unit normal on Γ oriented towards the exterior of Ω. We have:

Theorem 2.3. *We suppose that* Ω *has a boundary of class* $\mathcal{C}^{1,1}$. *Then there exists a continuous linear mapping* $\gamma_1 : H^2(\Omega) \to L^2(\Gamma)$, *such that for all* $v \in D(\bar{\Omega}), \gamma_1(v) = \frac{\partial v}{\partial \nu}\big|_\Gamma$. *The image of the space* $H^2(\Omega)$ *by the mapping* γ_0 *is denoted by* $H^{3/2}(\Gamma)$; *it is a Hilbert space with the norm defined by:*

$$\|g\|_{H^{3/2}(\Gamma)} = \inf_{v \in H^2(\Omega), \gamma_0(v) = g} \|v\|_{H^2(\Omega)}. \tag{2.3}$$

The kernel of the continuous mapping $\gamma = (\gamma_0, \gamma_1) : H^2(\Omega) \to L^2(\Gamma) \times L^2(\Gamma)$ *is the space* $H_0^2(\Omega)$.

Still using the density of regular functions, we can generalize Green's formula (1.3) for $v \in H^2(\Omega)$ and $v \in H^1(\Omega)$ in the following sense (Ω is bounded with a Lipschitz-continuous boundary):

$$\int_\Omega (\Delta uv)(x)dx = -\int_\Omega (\nabla u \cdot \nabla v)(x)\,dx + \int_\Gamma (\gamma_1 u \gamma_0 v)(x) d\Gamma(x). \tag{2.4}$$

Last, we can further weaken the hypothesis on u: the formula remains valid if $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ [4], but then the integral over Γ has to be replaced by a duality bracket, that we simply denote by $< .,. >_\Gamma$, between the space $H^{1/2}(\Gamma)$ and its dual space $H^{-1/2}(\Gamma)$ (in fact, if $u \in H^1(\Omega)$, one only has $\gamma_1 u \in H^{-1/2}(\Gamma)$). We get the following generalized green formula:

Proposition 2.4. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with a Lipschitz-continuous boundary. Then, for all* $u, v \in H^1(\Omega)$ *with* $\Delta u \in L^2(\Omega)$, *we have:*

$$\int_\Omega (\Delta uv)(x)dx = -\int_\Omega (\nabla u \cdot \nabla v)(x)dx + < \gamma_1 u, \gamma_0 v >_\Gamma. \tag{2.5}$$

2.2. The Variational Formulation

Let us go back to the example of the introduction. We can now show the equivalence between the boundary value problem and the variational one.

Proposition 2.5 *Let us suppose that* Ω *is a bounded open subset of* $\mathbb{R}^n$ *with a Lipschitz-continuous boundary and that* f *belongs to* $L^2(\Omega)$. *Let* $u \in H$ *with* $H = H_0^1(\Omega)$. *If* u *is a* *solution of Eqs.(1.4)-(1.5), then* $\Delta u \in L^2(\Omega)$ *and it satisfies the boundary value problem (1.1)-(1.2) (a.e. respectively on* Ω *and on* Γ*). Conversely, if* u *satisfies (1.1)-(1.2) with* $\Delta u \in L^2(\Omega)$, *then* u *is a solution of Eqs (1.4)-(1.5).*

Proof. If u is the solution of (1.1) with $\Delta u \in L^2(\Omega)$, we proceed as in Section 1.2. We multiply Equation (1.1) by $v \in H_0^1(\Omega)$, integrate over Ω and use Green's formula (2.5); since $\gamma_0 v = 0$, we get (1.4)-(1.5). Let us show the converse. For this, we only have to suppose $u \in H_0^1(\Omega)$. If Eqs (1.4)-(1.5) hold for any function v in $H_0^1(\Omega)$, they hold in particular for any function in $D(\Omega)$. We can interpret Equation (1.4) in the distributional sense; in fact, we get $< -\Delta u - f, v > = 0$, for any $v \in D(\Omega)$, which gives: $-\Delta u = f$ in $D'(\Omega)$. Now, since $f \in L^2(\Omega)$, we first get the following regularity result: $\Delta u \in L^2(\Omega)$ and $-\Delta u = f$ in $L^2(\Omega)$, so that Equation (1.1) is satisfied a.e. on Ω. Finally, we have $\gamma_0 u = 0$, because $u \in H_0^1(\Omega)$ and Equation (1.2) occurs a.e. on Γ.

It now remains to show that the variational problem (1.4)-(1.5), with $H = H_0^1(\Omega)$, has a unique solution. In this example, the answer is straightforward. Let us recall Riesz's theorem:

Theorem 2.6 (Riesz**).** *Let* H *be a Hilbert space, with scalar product denoted by* $(.,.)_H$. *Then, for any linear continuous form* L *defined on* H, *there exists a unique* $u \in H$, *such that: for all* $v \in H$, $L(v) = (u,v)_H$.

Now, according to Proposition 2.1, the Sobolev space $H_0^1(\Omega)$ is a Hilbert space for the reduced norm $|.|_{1,\Omega}$; application of Riesz's theorem then trivially gives:

Proposition 2.7. *The variational problem (1.4)-(1.5), with* $f \in L^2(\Omega)$, *has a unique solution in* $H_0^1(\Omega)$.

The model problem (1.1)-(1.2) has been completely solved. In order to obtain similar results for more general situations, we need a generalization of Riesz's theorem: this is the Lax-Milgram theorem.

Before proceeding further, let us give some useful regularity results for the solution of the variational problem. We have [5]:

Proposition 2.8. *Let* u *be the solution of the variational problem (1.4)-(1.5), with* $f \in L^2(\Omega)$ *given by Proposition 2.7. If the boundary* Γ *is of class* $\mathcal{C}^{1,1}$, *then:* $u \in H^2(\Omega)$.

Remark 2.9. This regularity result can be generalized. If $f \in H^k(\Omega)$, with $k \geq 0$, then, under suitable assumptions on the regularity of the boundary, we shall have: $u \in H^k(\Omega)$. Moreover, this result is valid for more general elliptic operators (we shall see examples in Section 3 below), as long as the coefficients of these operators are sufficiently smooth.

There are in fact two types of regularity results:

(1) a "local" regularity result, which only depends on the regularity of the coefficients of the elliptic operator involved; the result is:

if $f \in H^k_{loc}(\Omega)$, then $u \in H^{k+2}_{loc}(\Omega)$,

where:

$$H^k_{loc}(\Omega) = \{ f \in D'(\Omega), \text{ such that for all } \varphi \in D(\Omega), f\varphi \in H^k(\Omega)\}.$$

(2) a "global" regularity result, i.e. up to the boundary. For this, the regularity of the boundary is essential, such as the type of boundary conditions. We refer for example to [3] for a counter example, when the domain is not smooth enough.

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