FUNDAMENTALS OF OPERATIONS RESEARCH

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Summary

This article provides an overview of the fundamental classes of models or mathematical

programs used in Operations Research (OR): the *standard problems* that have been developed in mathematical Operations Research (MOR). We identify model characteristics, outline fundamental solution methods, identify areas of application for linear, integer and nonlinear models, and discuss aspects of implementation.

1. Introduction

Models and algorithms are the central concepts that distinguish Operations Research (OR) from other decision-making approaches. In other words, all Operations Research processes involve formulating, analyzing, and manipulating mathematical models. Therefore the resolution of a decision problem using OR is based on two abilities:

- The analyst's ability to translate the specific decision problem communicated by the problem owner, a manager for instance, into a mathematical model that can be used to compare the extent to which each alternative solution satisfies the manager's objective.
- The mathematician's ability to solve the formal mathematical model efficiently using an algorithm, usually employing computer software.

Mathematical decision models or optimization models have three components: decision variables, uncontrollable variables and parameters (model data), and output variables. These components are related to each other by mathematical relationships, the so-called "model logic."

The decision variables describe alternative courses of action that may be considered. In an investment problem, decision variables may represent the amount of money to be invested in each asset; in a production-scheduling problem they might represent, for example, the assignment of a certain machine and of production time to a specific job. Decision variables are classified as independent variables, and it is part of the formal or mathematical problem to find the best possible values for them.

In any decision situation there are factors that affect the output but are not under the control of the decision maker: for example (market) prices for assets, or the availability of machine resources. These uncontrollable variables and parameters place limits on the decision maker's courses of action. These variables are also classified as independent and they result in formal constraints being placed on the values for the decision variables.

Output variables reflect the level of effectiveness of a certain course of action, indicating how well the decision maker attains his or her goals. In an investment problem an output variable may measure the return on investment or capital value; in a production-scheduling problem an output variable may quantify the average tardiness of jobs.

The logic of the optimization model links variables and parameters using sets of mathematical expressions, such as functional equations or inequalities. Here we distinguish two conceptual classes of functions: *objective functions* and *constraints*. An objective function measures the utility of the decision, i.e. the course of action represented by a particular quantification of the decision variables. A constraint restricts the choice of values for the decision variables to meaningful and allowable ones: in other words, values that allow the courses of action represented by particular quantifications of the decision variables to be implemented in the real decision situation.

In this way, OR modeling formalizes the decision situation as the task of "optimizing an objective function subject to constraints," and an *optimization model* has the following general form:

$$\begin{aligned} &\text{Maximize} \quad f(x_1,...,x_n) && \text{subject to} \\ & g_i(x_1,...,x_n) \le 0 && \text{for} \quad i=1,...,n \end{aligned} \tag{1}$$

Model (1) is a formal mathematical problem; in mathematics this type of problem is referred to as a *mathematical programming problem* or *mathematical program* (MP). Note that we prefer the term "model" when focusing on the properties of a formal construct representing a certain decision problem. We would prefer the term "program" when discussing formal mathematical aspects of (1): for instance, the development of solution procedures.

Mathematical programs or optimization models may be divided into different classes, with programs/models in the same class sharing specific formal criteria:

- With respect to the quality of information (data) we distinguish *deterministic models* and *stochastic models*,
- With respect to the objective we distinguish *single-objective* and *multiple-objective* models,
- With respect to the domain of the decision variables we distinguish *continuous* and *discrete models*, and
- With respect to the type of objective function and constraints we distinguish *linear* and *nonlinear models*.

If all of the uncontrollable variables of the model are known and cannot vary, then the model is termed a deterministic model; if any of the uncontrollable inputs are uncertain, it is referred to as a stochastic model. Market demand in production planning models is generally uncertain. Stochastic models are more difficult to analyze since the value of the output (i.e. the value of the objective function) cannot be determined even if all values for the controllable variables are fixed. The conceptual foundation of stochastic models, as well as algorithmic aspects and applications, are discussed as a specific topic (see *Stochastic Operations Research*).

A crucial step in mathematical modeling is the formulation of the objective function, which requires the development of a quantitative measure of effectiveness with regard to each (managerial) objective. Sometimes more than one objective has been formulated: for instance, due to the involvement of different managers in the analysis representing different departmental views (perhaps a "production view" and a "marketing view"). In these situations it is necessary to combine these measures. A composite measure could be a higher organizational goal, or some "abstract" measure of overall utility.

Alternatively, the model may explicitly consider multiple objectives simultaneously. Specific concepts for analyzing preferences and measuring utility, as well as techniques for dealing with multiple objectives, are discussed separately (*see Decision Analysis*; *Multicriterion Decision Making*).

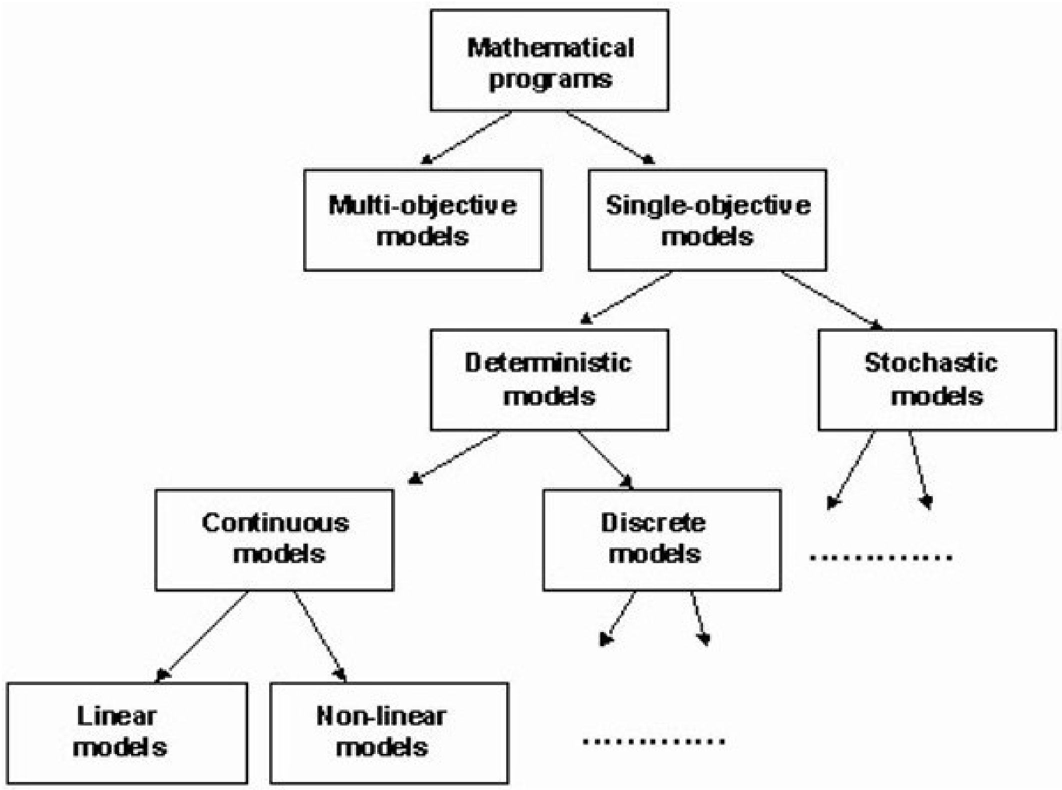

Figure 1. Classification of mathematical programs

The process of modeling and the relationship between problem, model, and algorithm are described in the theme level article (see *Optimization and Operations Research*). The analyst's challenge in this process is also described in the first topic article focusing on fundamentals of OR (see *The Role of Modeling*). The specific concepts behind OR software and its requirements are discussed in the last article under this topic (see *The Role of Software in Optimization and Operations Research*). The "mathematical" fundamentals of Operations Research - important special classes of mathematical programs/optimization models and related algorithmic concepts - are set out in the remaining articles under this theme (see *Linear Programming*, *Nonlinear Programming*, *Dynamic Programming*, *Discrete Optimization*).

This chapter provides an overview of the fundamental classes of OR-models or mathematical programs used in Operations Research (OR): the *standard problems* that have been developed in mathematical Operations Research (MOR). We identify model characteristics, outline fundamental solution methods, identify areas of application for linear, integer and nonlinear models, and discuss aspects of implementation. It is important to note that the standard models and programs that are the focus of MOR are purely ideal models, without any correspondence to concrete decision problems. Within different application areas the same model may be interpreted differently. Enriched with such an interpretation, these models acquire meaning and come to represent parts of the domain-knowledge or theory of the application domain, yet they remain purely ideal. Only when enriched with situative (i.e. problem-specific) data replacing the model parameters does a model become "real," in the sense of representing a concrete decision problem.

2. Linear Programming

A *linear program* (LP) is a deterministic optimization problem in which the (single) objective function is linear in the decision variables and the constraints consist of linear (in-)equalities. The canonical form of a linear program is

$$\text{maximize} \quad c_1x_1 + \cdots + c_nx_n \tag{2}$$

$$\text{subject to} \quad \begin{array}{l} a_{1,1}x_1 + \cdots + a_{1,n}x_n\,(\leq,=,\text{or}\geq)\,b_1 \\ \vdots \end{array} \tag{3}$$

$$\begin{array}{r} a_{m,1}x_1 + \cdots + a_{m,n}x_n\,(\leq,=,\text{or}\geq)\,b_m \\ x_1 \geq 0,\ldots,x_n \geq 0 \end{array} \tag{4}$$

where m, $n \in \mathbb{N}$, c_j, b_i and $a_{i,j}$ are constants, x_j are the decision variables (i=1,…,m; j=1,…,n), and only one sign (≤, =, ≥) holds for each constraint in (3). The inequalities (4) are called *non-negativity conditions*. In all cases we assume that $m \leq n$.

In linear programming (and in optimization in general) the meaning of the term "solution" is quite different from its common application to the final answer of a problem. Here any specification of values for the decision variables is termed a solution, regardless of whether or not it is a desirable -or even an allowable- choice. A *feasible solution* is a solution for which all the constraints are satisfied, and an *optimal solution* is a feasible solution that has the most favorable value in the objective function.

In the following sections, we survey a number of important (idealistic) problems that lend themselves naturally to being modeled as linear programs.

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Biographical Sketch

Ulrich Derigs is director of the Department of Information Systems and Operations Research (WINFORS) at the University of Cologne, Germany. He received a master's degree (1975) and a doctoral degree (1979) in Mathematics, and a doctoral degree in Economics (1981), from the University of Cologne. In 1985 he completed his habilitation at the University of Bonn. Ulrich Derigs was research assistant at the Mathematical Institute (1976–1979), and at the Seminar for Industrial Engineering (1979–1981), at the University of Cologne. From 1981 to 1985 he was research assistant at the Institute for Econometrics and Operations Research at the University of Bonn and member of a *Sonderforschungsbereich* (DFG). From 1985 to 1990 he was Professor of Information Systems and Operations Research at the University of Bayreuth. Since 1990 he has been professor at the University of Cologne.

Ulrich Derigs has done extensive research in combinatorial optimization, with an emphasis on the design, analysis and evaluation of efficient algorithms and industrial applications. Today his interest lies in the interface between Operations Research and information systems. His focus is on the design and implementation of model-based decision support concepts and systems in different application areas, including routing and scheduling, production planning, logistics, finance, and telecommunication.

Ulrich Derigs is a member of the editorial board of several journals. From 1988 to 1992 he was editor-in-chief of *OR Spectrum*, the journal of the German OR society, DGOR. From 1992 to 1998 he was member of the board of DGOR, and from 1996 to 1998 its president.