COMPUTATIONAL METHODS FOR SEISMIC ANALYSIS OF STRUCTURES

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Contents

Summary

While nonlinear finite element models offer great flexibility in simulating the response of structural members to seismic forces, computational methods are required at the global level in order to ensure that a numerical solution is found at each time step in dynamic analysis. This chapter begins with a generic representation of externally applied loads in static and dynamic seismic analyses of structures. Then, algorithms used to find equilibrium solutions at each time-step of analysis are discussed. It concludes with time integration methods that advance seismic analysis from one time step to the next, which are then discussed for static pushover and nonlinear dynamic analyses. Pseudo-code for implementation is presented in order to demonstrate the computational steps taken by each time integration method and equilibrium solution algorithm. Although other computational methods of seismic analysis are available, the focus of this chapter is on "OpenSees", an open source finite element software framework, which is available for free download.

1. Representation of Loads in Seismic Analysis

In the modeling of structures with N externally applied loads for seismic analysis, the set of applied loads is denoted as an N-vector of time-dependent forces (pseudo-time for static analysis)

$$\mathbf{p}(t) = \sum_{i=1}^{N} \lambda_i(t)\, \mathbf{p}_{\mathrm{ref}_i} \,, \tag{1}$$

where $\lambda_i(t)$ is the time-varying load factor and $\mathbf{p}_{\mathrm{ref}_i}$ is the reference load vector for the i-th component of the N-vector of load patterns applied to the structural model. The load patterns usually represent gravity loads, lateral loads, and earthquake excitation, while time-dependence of the i-th pattern is embodied by $\lambda_i(t)$.

2. Equilibrium Solution Algorithms

Under seismic loading, it is anticipated that structural members will yield in order to dissipate energy and reduce member force demands. Yielding will cause a change in member stiffness, thereby requiring an iterative algorithm to find structural equilibrium. For arbitrary member force-deformation response, root-finding algorithms enable an implicit solution to structural equilibrium and are more versatile and reliable than event-to-event and explicit solution algorithms.

Each of the following equilibrium solution algorithms seeks the nodal displacements $\mathbf{u}$ that satisfy the residual equilibrium equation at discrete time steps t_n, of a seismic simulation

$$\mathbf{r}\big(\mathbf{u}(t_n)\big) = \mathbf{p}(t_n) - \mathbf{p}_{\mathrm{r}}\big(\mathbf{u}(t_n)\big) = \mathbf{0}\,, \tag{2}$$

where $\mathbf{p}$ is the time-varying external load vector (discussed later in this chapter) and $\mathbf{p}_{\mathrm{r}}$ is the vector of structural resisting forces assembled from element contributions by standard finite element procedures. The nodal displacements at a given time step t_n, of a seismic analysis are found by iteration

$$\mathbf{u}_n^{j+1} = \mathbf{u}_n^{j} + \Delta\mathbf{u}^{j+1}, \tag{3}$$

where the superscript $j = 0,1,2,\ldots$ denotes the number of iterations performed until an acceptable equilibrium solution is found at the time step t_n or until a tolerable limit on j is set for the search to end the search. The time-dependence indicator (t_n) of nodal displacements meaning “at time step t_n“, is implied by writing $\mathbf{u}(t_n)$ for simplicity as $\mathbf{u}_n$, throughout this chapter. At each stage of iteration j, a set of trial nodal displacements are computed from which the state of the structure is determined by nonlinear finite element procedures and then checked for satisfaction of the equilibrium equations.

2.1. Newton Algorithms

The most general approach to solving equilibrium equations for nonlinear structural response is the Newton-Raphson method, and its variants including quasi- and accelerated Newton methods. The Newton family of algorithms attempt to find, for a given load vector $\mathbf{p}$, the nodal displacements $\mathbf{u}$, that make the residual equilibrium vector equal to zero. This equilibrium equation is for nonlinear static analysis. Extensions of iterative root finding algorithms for nonlinear dynamic analysis are discussed in conjunction with time integration methods later in this chapter.

2.1.1. Newton-Raphson Algorithm

The Newton-Raphson algorithm is frequently used in nonlinear structural analysis due to its fast convergence. A first order approximation of the residual vector in Eq. (2) leads to the Newton-Raphson iteration

$$\mathbf{r}^{j+1} = \mathbf{r}^{j} + \left.\frac{\partial \mathbf{r}}{\partial \mathbf{u}}\right|_{\mathbf{u}^j} \Delta\mathbf{u}^{j+1} . \tag{4}$$

By assuming that the residual vector $\mathbf{r}^{j+1}$ is zero at iteration $j+1$, the following linear system of equations is obtained for the displacement increment

$$\left(\left.\frac{\partial \mathbf{r}}{\partial \mathbf{u}}\right|_{\mathbf{u}^j} \right) \Delta\mathbf{u}^{j+1} = -\mathbf{r}^{j} . \tag{5}$$

The partial derivative of the residual force vector of Eq. (2) leads to the tangent stiffness matrix of the structural model

$$\mathbf{K}_{\mathrm{T}}^{j} = \left.\frac{\partial \mathbf{p}_{\mathrm{r}}}{\partial \mathbf{u}}\right|_{\mathbf{u}^j} , \tag{6}$$

which is assembled by standard nonlinear finite element procedures from the element stiffness matrices evaluated at $\mathbf{u}^j$. The displacement increment at each stage of iteration in a time step is then found by solution to the linear system of equations

$$\mathbf{K}_{\mathrm{T}}^{j} \Delta\mathbf{u}^{j+1} = \mathbf{r}^{j} . \tag{7}$$

With an initial displacement vector $\mathbf{u}^0$, the Newton-Raphson iteration is performed by repeated evaluation of the residual and tangent stiffness until equilibrium is achieved (norm of the residual vector is less than a specified tolerance) or the specified tolerable number of iterations ($j_{\max}$) is reached, as shown in Figure 1 and the following pseudo-code.

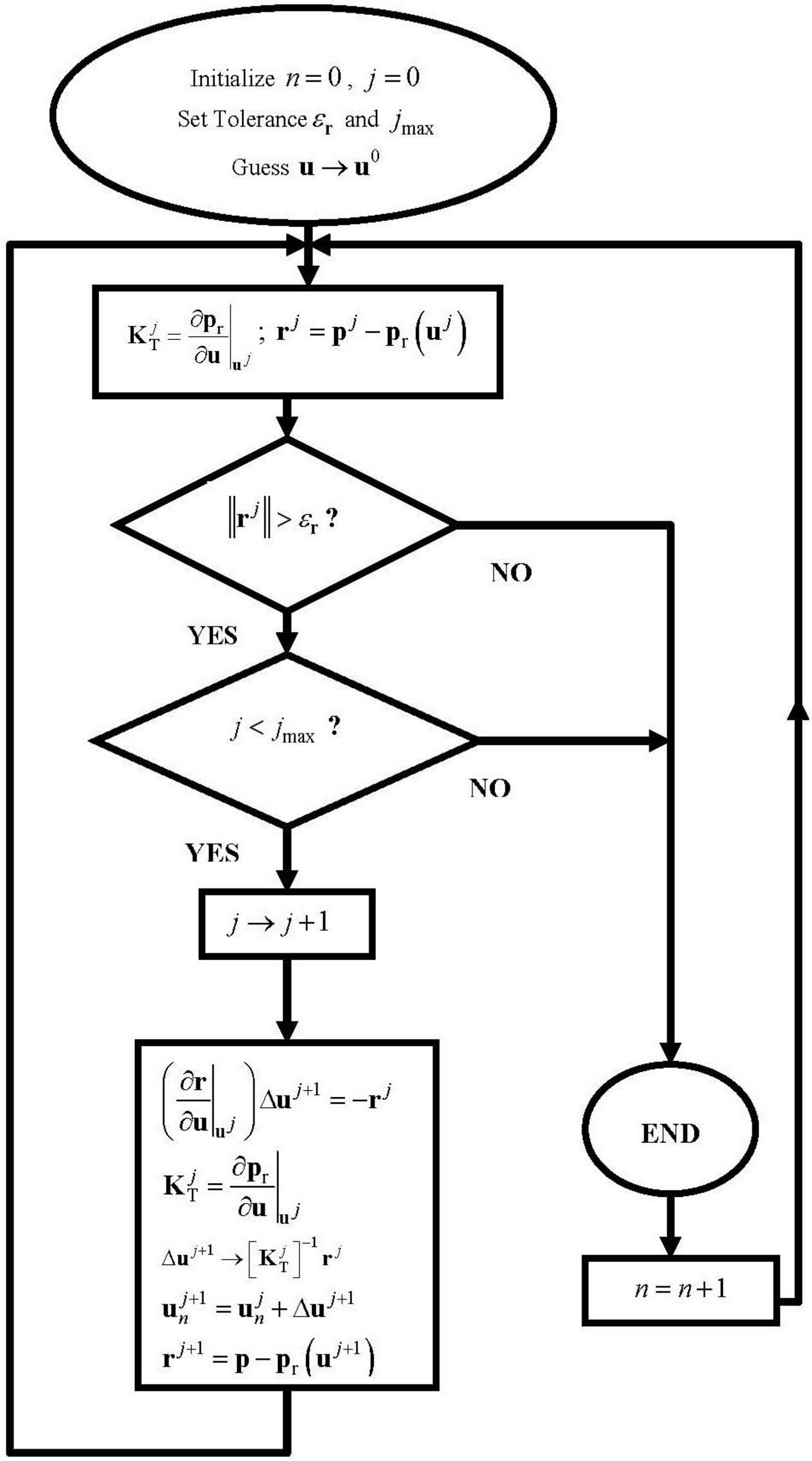

Figure 1. Newton-Raphson algorithm.

The advantage of the Newton-Raphson algorithm is its quadratic convergence near the exact equilibrium solution where the error at the current iteration is less than the square of the error at the previous iteration

$$\left\|\mathbf{u}^{j+1}-\mathbf{u}^{\text{exact}}\right\| \leq C\left\|\mathbf{u}^{j+1}-\mathbf{u}^{\text{exact}}\right\|^2. \tag{8}$$

The constant C depends on the first and second derivatives of the residual near the exact solution. The main drawback to the Newton-Raphson algorithm is the computational expense of repeatedly forming and factorizing the tangent stiffness matrix at each stage of iteration. This can represent a significant amount of computation for large structural models with thousands of equilibrium equations.

Pseudo-code for the Newton-Raphson algorithm

```
U = U0; % Initial guess
R = Pf - Pr(U); % Initial residual
jMax given
j = 0; % Number of iterations
while norm(R) > tol && j < jMax
     assemble KT = Pr'(U)
     solve dU = KT \ R
     update U = U + dU
     assemble R = Pf - Pr(U)
     j = j + 1;
end
```

2.1.2. Modified Newton Algorithm

In an attempt to reduce the computational burden of the Newton-Raphson algorithm, the tangent stiffness matrix at the first iteration of a time step can be held constant over all subsequent iterations in that time step. All computational steps for this “Modified” Newton algorithm stay the same, except for that the tangent stiffness matrix assembly and its factorization are moved out of the iteration loop, as shown in Figure 2 and the following pseudo-code.

This modification of the Newton algorithm can lead to a significant computational savings for large structural models where the tangent stiffness matrix does not change substantially during a time step. The drawback of this method is its linear convergence rate and the generally larger number of residual evaluations required to find equilibrium. A tradeoff between the Newton-Raphson and Modified Newton algorithms is achieved by updating the tangent stiffness periodically every m iterations at any time step [9]. The value of m depends on the relative cost of matrix factorization versus residual evaluation for the structural model at hand and generally gives a super-linear rate of convergence.

2.1.3. Other Newton Algorithms

Two broad classes of Newton-like methods have been developed: accelerated- and quasi-Newton methods. Accelerated Newton methods improve the convergence rate of the Modified Newton algorithm by performing low-cost matrix-vector operations, e.g., least squares problems in low-rank Krylov subspaces, at each stage of iteration. Quasi-Newton methods improve the Modified Newton convergence by altering the tangent stiffness matrix, or its factorization, at each stage of iteration in order to improve equilibrium search directions. The BFGS method is common in structural mechanics applications where the tangent stiffness matrix is symmetric.

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Biographical Sketch

Michael H. Scott is a Professor of Structural Engineering in the School of Civil and Construction Engineering at Oregon State University. He completed his Ph.D. in Structural Engineering at the University of California, Berkeley in 2004. Prof. Scott is one of the core developers of OpenSees, the Open System for Earthquake Engineering Simulation, a widely used open source framework for nonlinear finite element analysis.