THERMOELASTICITY

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Keywords: Thermodynamics, thermoelastic materials, constitutive equations, thermal effects, uniqueness, variational theorems, waves, beams, explicit solutions.

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Summary

This chapter is concerned mainly with the basic problems of the linear theory of thermoelasticity. Beginning with the basic laws of thermodynamics, there follows a treatment of the constitutive equations and the derivation of the equations of nonlinear thermoelasticity. The next part of this work is devoted to the linear thermoelastodynamics. First, some basic theorems are established. Then, an investigation of thermoelastic waves is presented. The work concludes with a study of the theory of thermoelastic equilibrium. Relevant examples which illustrate the theory are given throughout the text.

1. Introduction

The theory of thermoelasticity is concerned with the interaction between thermal field and the elastic bodies. The study of thermoelasticity was begun by Duhamel (1837) and Neumann (1885) who postulated the equations of the linear thermoelasticity for isotropic bodies. These equations have been justified by Biot (1956) on the basis of irreversible thermodynamics. A derivation founded on modern continuum
thermodynamics has been given by Eringen (1967). The theory of thermoelasticity is of great importance. The published work in thermoelasticity is so large that it is not possible to do justice to all contributors by mere mention of their names. An account of the historical development, as well as references to various contributions, may be found in the monographs by Green and Adkins (1960), Boley and Weiner (1960), Truesdell and Toupin (1960), Nowacki (1962), Carlson (1972), Day (1985), Ieşan and Scalia (1996).

In this work we present a short account of the linear theory of thermoelasticity. The exposition of nonlinear thermoelasticity is presented to provide a base for the linear theory. The reader interested in the nonlinear thermoelasticity is referred to the books by Racke and Jiang (2000) and Ieşan and Scalia (1996).

The present work consists of three main parts. In the first part (Sections 2-5) we focus attention to the derivation of the equations of thermoelasticity. The second part of this article (Sections 6-10) contains a study of the dynamic theory of thermoelasticity. In the last part we investigate some problems of the theory of thermoelastostatics.

To review the vast literature on applications and special problems is not our intention; considerations of space and time have caused extensive selection to be made. The illustrations included are examples considered relevant to the purpose of the text.

The assumptions of zero initial stress and uniform reference temperature are crucial to the development of the classical linear thermoelasticity. Thermoelasticity of bodies with initial stresses and non-uniform reference temperature is not considered here. The reader interested in these subjects will find a full account in the works of Knops and Wilkes (1973) and Ieşan and Scalia (1996).

In recent years there has been some interest in thermoelasticity of polar materials and the theories of thermoelasticity with finite wave speeds. For an extensive review of the literature on these theories the reader is referred to the monographs by Nowacki (1986), Chandrasekharaiah (1986), Eringen (1990), Jou et al. (1996), Müller and Ruggeri (1998), Ieşan (2004). We make no claim to completeness. It is hoped that the present work gives an accessible treatment of a part of the contributions that have been made to the subject.

2. Preliminaries

In what follows we consider a body that at time $t_0$ occupies the region $B$ of Euclidean three-dimensional space $E^3$. We assume, unless specified otherwise, that $B$ is a bounded regular region.

The configuration of the body at time $t_0$ is taken as the reference configuration. The motion of the body is referred to the reference configuration and a fixed system of rectangular Cartesian axes. We identify a typical particle $x$ of the body with its position $\mathbf{x}$ in the reference configuration. The coordinates of a typical particle $x$ in $B$ are
 continuity, mechanics, thermomechanics, stress, strain, thermal expansion, material, continuity, time, interval, function, partial derivative, determinant, configuration, present, configuration, summation, differentiation, convention, subscripts, range, Cartesian, coordinate, material derivative, superposed dot, superindex, boldface, tensors, order, mass density, body force, Piola-Kirchhoff stress tensor, conservation law, linear momentum, first Piola-Kirchhoff stress tensor, second Piola-Kirchhoff stress tensor.

The local form of the conservation law of linear momentum can be expressed as

$$T_{ji,j} + \rho_0 f_i = \rho_0 \ddot{y}_i \quad \text{on} \quad B \times (t_0, t_1),$$

where $T_{ji}$ is the first Piola-Kirchhoff stress tensor, $\rho_0$ is the mass density at time $t_0$ and $f_i$ is the body force per unit mass.

If we define the second Piola-Kirchhoff stress tensor $S_{ij}$ by

$$T_{ki} = y_{i,j} S_{kj},$$

then the local form of the conservation law of moment of momentum reduces to
\[ S_{ij} = S_{ji}. \quad (5) \]

We denote by \( E_{ij} \) the Lagrangian strain tensor,

\[ E_{ij} = \frac{1}{2} (y_{k,i} y_{k,j} - \delta_{ij}) \text{ on } B \times (t_0, t_1), \quad (6) \]

where \( \delta_{ij} \) is Kronecker’s delta.

The local form of the first law of thermodynamics can be written as

\[ \rho_0 \dot{e} = S_{ij} \dot{E}_{ij} + \rho_0 S + Q_{j,i} \text{ on } B \times (t_0, t_1), \quad (7) \]

where \( e \) is the internal energy per unit mass, \( S \) is the heat supply per unit mass, and \( Q_j \) is the heat flux associated with surfaces in \( B' \) which were originally coordinate planes perpendicular to the \( x_j \)-axes through the point \( x \), measured per unit undeformed area.

We assume that \( f_i \) and \( S \) are continuous on \( B \times (t_0, t_1) \), \( T_{ij} \) and \( Q_j \) are of class \( C^{1,0} \) on \( B \times (t_0, t_1) \) and continuous on \( \overline{B} \times [t_0, t_1] \).

Let \( P \) be a region of the continuum bounded by a surface \( \partial P \) at time \( t \), and suppose that \( P \) is the corresponding region at time \( t_0 \), bounded by the surface \( \partial P \). We denote by \( n_i \) the components of the outward unit normal at \( \partial P \). Let \( \mathbf{t} \) be the stress vector associated with the surface \( \partial P \), but measured per unit area of the surface \( \partial P \), and let \( q \) be the heat flux across the surface \( \partial P \), measured per unit area of \( \partial P \). Then, we have

\[ t_i = T_{ji} n_j, \quad q = Q_{j,i} n_j. \quad (8) \]

We denote by \( T \) the absolute temperature, which is assumed to be positive. Let \( \eta \) be the entropy per unit mass. We assume that \( \eta \) is of class \( C^{0,1} \) and \( T \) is of class \( C^{2,1} \) on \( B \times (t_0, t_1) \). The local form of the second law of thermodynamics can be expressed as

\[ \rho_0 T \dot{\eta} - \rho_0 S - Q_{j,i} + \frac{1}{T} Q_j T_{j,i} \geq 0. \quad (9) \]

If we introduce the Helmholtz free-energy,

\[ \psi = e - T \eta, \quad (10) \]
then the equation of energy can be written in the form

\[ \rho_0 (\dot{\psi} + \dot{T} \eta + T \dot{\eta}) = S_{ij} \dot{E}_{ij} + Q_{j,j} + \rho_0 S. \]  

(11)

From (9) and (11) we obtain the following local dissipation inequality

\[ S_{ij} \dot{E}_{ij} - \rho_0 (\dot{\psi} + \dot{T} \eta) + \frac{1}{T} Q_j T_{j,j} \geq 0. \]  

(12)

3. Constitutive Equations

A thermoelastic material is defined as one for which the following constitutive equations hold

\[ \psi = \hat{\psi}(E_{mn}, T, T_k, x_r), \]

\[ S_{ij} = \hat{S}_{ij}(E_{mn}, T, T_k, x_r), \]

\[ \eta = \hat{\eta}(E_{mn}, T, T_k, x_r), \]

\[ Q_i = \hat{Q}_i(E_{mn}, T, T_k, x_r). \]

(13)

We suppose that the functions \( \hat{\psi}, \hat{S}_{ij}, \hat{\eta} \) and \( \hat{Q}_i \) are of class \( C^1 \) on their domain. In the case of homogeneous bodies the constitutive functions do not depend on \( x_r \). Clearly, the constitutive equations (13) satisfy the principle of material frame-indifference.

Let us study the restrictions placed on the constitutive functions by the second law. We introduce the notation

\[ \sigma = \rho_0 \dot{\psi}. \]  

(14)

In view of (13), the inequality (12) becomes

\[ \left( S_{ij} - \frac{\partial \sigma}{\partial E_{ij}} \right) \dot{E}_{ij} - \left( \rho_0 \eta + \frac{\partial \sigma}{\partial T} \right) \dot{T} - \frac{\partial \sigma}{\partial T_{j,j}} \dot{T}_{j,j} + \frac{1}{T} Q_j T_{j,j} \geq 0. \]  

(15)

We assume that \( \sigma \) in (15) is arranged as a symmetric function of \( E_{ij} \). For a given deformation and temperature, the inequality (15) is valid for all arbitrary values of \( \dot{E}_{ij}, \dot{T} \) and \( \dot{T}_{j,j} \), subject to \( E_{ij} = E_{ji} \). Thus, in absence of internal constraints, from (15) we obtain (see Coleman and Mizel (1964), Carlson (1972))
\[ S_{ij} = \frac{\partial \sigma}{\partial E_{ij}}, \quad \rho_0 \eta = -\frac{\partial \sigma}{\partial T}, \quad \frac{\partial \sigma}{\partial T_i} = 0, \]

and

\[ Q_j T_{j} \geq 0. \quad (16) \]

We conclude that the constitutive equations of thermoelastic bodies are given by

\[ \sigma = \hat{\sigma}(E_{ij}, T, x_k), \]
\[ S_{ij} = \frac{\partial \sigma}{\partial E_{ij}}, \quad \rho_0 \eta = -\frac{\partial \sigma}{\partial T}, \]
\[ Q_p = \hat{Q}_p(E_{ij}, T, T_k, x_m). \quad (17) \]

In view of (17), the energy equation (11) takes the form

\[ \rho_0 T \eta = Q_{j,i} + \rho_0 S \quad \text{on } B \times (t_0, t_1). \quad (18) \]

The next result is a consequence of inequality (16).

**Theorem 3.1.** The heat flux vanishes whenever the temperature gradient vanishes,

\[ \hat{Q}_i(E_{mn}, T, 0, x_k) = 0. \quad (19) \]

**Proof.** Let us consider the function

\[ f(\xi_1, \xi_2, \xi_3) = \xi_i \hat{Q}_i(E_{mn}, T, \xi_1, \xi_2, \xi_3, x_k), \]

where \( E_{mn}, T \) and \( x_k \) are fixed. The inequality (16) shows that \( f \) is nonnegative. Since \( f(0, 0, 0) = 0 \), the function \( f \) has an extremum at \((0, 0, 0)\). If we impose that \( \frac{\partial f}{\partial \xi_k} = 0 \) at \((0, 0, 0)\), then we obtain the desired result. □

This theorem has been established by Pipkin and Rivlin (1958).

**4. Equations of the Nonlinear Thermoelasticity**

The basic equations of the nonlinear theory of thermoelasticity consist of equations of motion (3), the energy equation (18), the constitutive equations (17) and the geometrical equations (6), on \( B \times (t_0, t_1) \), where \( t_1 \) is some time instant that may be infinite. The functions \( \rho_0, f_i \) and \( S \), and the constitutive functionals \( \hat{\sigma} \) and \( \hat{Q}_j \) are prescribed. The response functionals \( \hat{Q}_j \) are subjected to the restriction (16). To the field equations
we must adjoin boundary conditions and initial conditions. In the case of the mixed boundary-value problem the boundary conditions are

\[
y_i = \tilde{y}_i \quad \text{on } S_1 \times (t_0, t_1), \quad T = \tilde{T} \quad \text{on } S_3 \times (t_0, t_1),
\]
\[
T_{ji} n_j = \tilde{t}_i \quad \text{on } S_2 \times (t_0, t_1), \quad Q_i n_i = \tilde{q} \quad \text{on } S_4 \times (t_0, t_1),
\]

where \( S_i, (i = 1, 2, 3, 4) \), are sub-surfaces of \( \partial B \) such that \( S_1 \cup S_2 = S_3 \cup S_4 = \partial B, S_1 \cap S_2 = S_3 \cap S_4 = \emptyset \), and \( \tilde{y}_i, \tilde{T}, \tilde{t}_i \) and \( \tilde{q} \) are prescribed functions. The initial conditions are

\[
y(x, 0) = y^0(x), \quad \dot{y}(x, 0) = v^0(x), \quad \eta(x, 0) = \eta^0(x), \quad x \in \bar{B},
\]

where \( y^0, v^0 \) and \( \eta^0 \) are given. We assume that

(i) \( \rho_0 \) is continuous and strictly positive on \( \bar{B} \);
(ii) \( f \) and \( S \) are continuous on \( \bar{B} \times [t_0, t_1] \);
(iii) \( y^0, v^0 \) and \( \eta^0 \) are continuous on \( \bar{B} \times [t_0, t_1] \);
(iv) \( \tilde{y}_i \) are continuous on \( S_1 \times [t_0, t_1] \) and \( \tilde{T} \) is continuous on \( S_3 \times [t_0, t_1] \);
(v) \( \tilde{t}_i \) are continuous in time and piecewise regular on \( S_2 \times [t_0, t_1] \) and \( \tilde{q} \) is continuous in time and piecewise regular on \( S_4 \times [t_0, t_1] \).

The mixed problem of thermoelastodynamics consists in finding the functions \( y_i \) of class \( C^2 \) and \( T \) of class \( C^{2,1} \) on \( B \times (t_0, t_1) \) that satisfy Eqs. (3), (18), (17) and (6) on \( B \times (t_0, t_1) \), the boundary conditions (20) and the initial conditions (21).

It is possible to set up more complicated boundary conditions than those considered here. In the case of the convection condition on the boundary, the thermal condition is

\[
Q_i n_j = h(T - T_e) \quad \text{on } \partial B \times (t_0, t_1).
\]

Here \( T_e \) is the temperature of surrounding medium and \( h \) is the heat transfer coefficient.

The exposition of nonlinear thermoelasticity given here is presented to provide a base for the linear theory. The reader interested in the nonlinear thermoelasticity will find a full account in the books by Racke and Jiang (2000) and Ieșan and Scalia (1996).
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Biographical Sketches

Dorin Iesan is professor of Rational Mechanics at the Faculty of Mathematics, “Al. I. Cuza” University of Iasi, Romania. His research activity is devoted to the mathematical problems of Mechanics of Continua. D. Iesan is the author of 137 papers and seven books concerned with various problems of elasticity, thermoelasticity and viscoelasticity. He was a visiting professor at the Royal Institute of Technology from Stockholm, Institut d’Estudis Catalans of Barcelona and University of Catania, and presented conferences at various international seminars and universities.