APPLICATIONS TO FLUID MECHANICS: WATER WAVE PROPAGATION

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Summary

Fluid Mechanics is the discipline within the broad field of applied mechanics concerned with the behavior of fluids and gases in motion or at rest. As such it encompasses a vast array of problems that may vary from large scale geophysical flows to the very small scale study of blood flow in capillaries. Within the geophysical flows, the modeling of ocean waves is of great interest for several fields of science and engineering, i.e., oceanography, underwater acoustics, coastal and ocean engineering or navigation.

In this chapter we show that applied mechanics principles such as conservation of mass,
momentum and energy are the foundations of the mathematical modeling of wave propagation. Short wave (T<30 s) propagation is divided into phase-averaged and phase-resolving models. We show the main differences between both approaches by describing the equations and presenting modeling applications at a specific site.

1. Introduction

Coastal dynamics is mostly dominated by surface waves. Waves are disturbances of the equilibrium state in any body of material, in this case a fluid, which propagate through that body over distances and times much larger than the characteristic wave lengths and periods of the disturbances transferring energy and momentum. Everyone interested in ocean processes is aware of the fact that there is an important variety of waves occurring in the oceans and along our shores which can be classified according to their period, wave length or nature of the forcing generating the free surface disturbance.

Tides are the cyclic rising and falling of the ocean surface generated by the forces resulting from the interaction between the earth-moon-sun system. Tides’ main constituent periods range from 12 to 24 hours and their wave lengths accordingly vary from hundreds to a few thousands kilometers. With slightly shorter wave periods and wave lengths storm surges are large-scale offshore elevations of the ocean surface associate with a low atmospheric pressure system and high wind speeds. The space and time scales of storm surges are closely correlated with those of the generating storm, i.e. in the order of hundreds of kilometers and one or two days. The combined effect of low pressure, persistent winds and shallow bathymetry may result in catastrophic flooding due to the piling of water against the shore. Even more devastating can become Tsunami waves, a series of waves smaller in wave length and period than storm surges and created when a body of water is rapidly displaced due to earthquakes, land or submarine slide, volcanic eruptions or explosions. The 2004 Indian Ocean Tsunami was generated by an undersea earthquake which triggered a series of devastating tsunamis along the shores of Indonesia, Sri Lanka, India, Thailand and other countries with waves up to 30 m and killing large numbers of people and inundating coastal communities. Tsunami periods may vary from tenth’s of minutes to hours and may reach a propagation speed (so-called celerity) up to 800 km/h.

In the range between 30 seconds and a few minutes energy in the ocean is present in the ocean in the form of infragravity waves. Infragravity waves are generated by groups of wind-generated waves and are very relevant in the surf zone where they may dominate beach run-up or be responsible for part of the sediment transport. Furthermore, infragravity waves are the main forcing mechanism of harbor resonance.

Wind generated waves are dominant in the ocean. Usually known as gravity waves, they have wave periods shorter than 30 s and wave lengths in the order of hundreds of meters.

When wind waves are generated by a distance storm, they usually consist of a wide range of wave frequencies and can be described as irregular and short-crested. The wave component with a higher wave frequency propagates at a slower speed than those with lower wave frequencies. As they propagate across the continental shelf towards the
coast, long waves lead the wave group and are followed by short waves. When they leave the generation area they become regular and long-crested. In the deep water, wind generated waves are not affected by the bathymetry. Upon entering shoaling waters, however, they are either refracted by bathymetry or current, or diffracted around abrupt bathymetric features such as submarine ridges or canyons. A part of wave energy is reflected back to the deep sea. Continuing their shoreward propagation, waves lose some of their energy through dissipation near the bottom. Nevertheless, each wave profile becomes steeper with increasing wave amplitude and decreasing wavelength. Because the wave speed is proportional to the square root of the water depth in very shallow water, the front face of a wave moves at a slower speed than the wave crest, causing the overturning motion of the wave crest. Such an overturning motion usually creates a jet of water, which falls near the base of the wave and generates a large splash. Turbulence associated with breaking waves is responsible not only for the energy dissipation but also for the sediment movement in the surf zone opening new disciplines such as sediment transport or coastal morphology.

Understanding and predicting the generation of waves and their transformation induced by coastal features or artificial structures are key issues in coastal science and engineering. In the following it will be shown that Fluid Mechanics is crucial for understanding and modeling waves.

In this chapter we will focus on wind waves but the same Fluid Mechanics principles apply to other types of waves previously described.

2. Classification of Wave Models

To date, two basic kinds of wave models can be distinguished: phase-resolving and phase-averaged models.

Even if wind waves are random in nature which may require a statistical approach, under certain circumstances waves can be described using a fully deterministic modeling based on hydrodynamics conservation laws, i.e. conservation of mass, momentum and energy. Using these equations a detailed description of the waves can be given by estimating free surface evolution, velocity and acceleration fields or wave-induced pressure which may be used to calculate forces and moments on structures. The application of phase-resolving numerical models, which require $10^{10}$ to $10^{100}$ time steps for each wave period, is still limited to relatively small areas, $O(1 \sim 10 \text{ km})$ and to relatively short periods of time (10 to hundreds wave periods), and is usually oriented towards the evaluation of wave propagation close to shore, wave agitation in harbors, wave and structure interaction or surf zone hydrodynamics. These models are also used to drive sediment transport and shore morphodynamics models.

For larger length scales, in the order of hundreds of kilometers and hours to year’s time scales the phase-resolving approach is not used since the amount of information required to describe waves would be overwhelming and computationally not affordable. Moreover, a detailed description of the kinematics and dynamics associated to waves is only necessary for certain practical applications at small scales (i.e. it’s not necessary to have a detailed resolution of wave kinematics in the North Atlantic to address a wave
agitation problem in a harbor in the North of Spain).

Phase-averaged models, based on the energy balance equation are more relax in the spatial resolution and can be used in much larger regions. The energy balance equation gives the rate of change of the sea state represented by the spectrum that is based on the idea that the profile of ocean waves can be evaluated as the superposition of a high number of harmonic waves, each with its own amplitude, frequency, wave length, direction and phase. The spectrum does represent statistical characteristics of the waves but does not provide detailed information of wave kinematics or dynamics. Moreover, the energy balance equation does include source functions of the generation of waves by wind and provides also a way to introduce wave dissipation due to several different mechanisms.

3. Phase-Averaged Models

The introduction of the concept of a wave spectrum was a key milestone in the development of wave modeling. The wave spectrum at a certain location describes the average sea state (time period of 15-30 min of actual waves for which statistical stationarity of wave characteristics is assumed) in a finite area around that location. The spectrum only gives information on the energy distribution of the waves and not on the phase of the individual waves. Therefore, models based on the energy balance equation, which are able to predict the spectrum at a certain location in the ocean taking into account effects of wave generation (by wind), wave-wave interaction and dissipation (by white-capping) are so-called phase-averaged models.

The spectral energy balance equation is given by

\[
\frac{\partial E(f, \theta, x, y, t)}{\partial t} + \frac{\partial C_{g,x} E(f, \theta, x, y, t)}{\partial x} + \frac{\partial C_{g,y} E(f, \theta, x, y, t)}{\partial y} = F(f, \theta; x, y, t) \quad (1)
\]

where the left-hand side of the equation represents the rate of change of the energy density of each wave component, \( E(f, \theta, x, y, t) \), \( C_{g,x} \) and \( C_{g,y} \) are the \( x \)- and \( y \)-components of the group velocity. \( F(f, \theta; x, y, t) \) is the so-called source term which includes the processes of wave generation by wind, wave-wave nonlinear interactions and dissipation by white-capping, a wave breaking process that occurs in deep water. For large-scale applications, this equation is usually transferred to a spherical coordinate system.
This so-called wave generation models are currently being used in operational systems and many times provide input information to run phase-resolving models in more limited areas. The most extended wave generation models are WAVEWATCH III (Tolman 1991, 1999) the WAM model (WAMDIG 1988, Komen et al. 1994).

Figure 1, shows the location of our applications in this chapter, the Port of Almeria. Almeria is located at the southeast part of Spain at the Mediterranean coast. The dominant and prevailing winds come from the southwest, and the area is characterized by a small tidal range of 0.6 m and a relatively strong wave climate. The water depths inside the harbor are in the range of 10 m. The harbor includes several breakwaters and quays.
Figure 2, shows the geometry of the harbor. As can be seen in this geometry the most offshore quay is protected by a detached breakwater to be constructed. Points P1 to P10, represent the locations where the input spectra at the offshore boundary of the phase-resolving numerical model, later explained in this chapter, are located. As will be shown, these spectra result from the wave propagation carried out using a phase-averaged model based on (1).

In order to evaluate how waves reach the Port of Almeria the wave climate transformation has to be evaluated. Depending on the track of the storms the wave may reach the harbor from different locations. Since the geographical domain is large a phase-average model (WAM) is used to obtain the wave climate in deep water. The model solving Eq. (1) allows the nesting of different meshes to increase spatial resolution while approaching the harbor. Using spectra provided by a west Mediterranean mesh (not shown) as an input for Mesh A, two additional meshes B and C, provide a high resolution information on wave spectra close to the harbor (50 m x 50 m grid). These meshes are used to run the SWAN (Simulating Waves Nearshore) model, a shallow water version of the wave generation model, WAM.
Figure 3. Location and size of the nested meshes used for the phase-averaged wave propagation model. Mesh A (500 m x 500m), Mesh B (100 m x100 m), Mesh C (50 m x 50 m)

Figure 4 a, b and c shows the results for given offshore wave conditions. The three plots show arrows indicating the direction of the incoming waves and a grey-scale indicating wave height.
Figure 4. Wave height and direction for wave propagation calculated using the phase-averaged model SWAN at the Port of Almeria. Results are shown for the nested meshes A, B and C and waves arriving from the southeast.
For the case presented waves arrive from the southeast perpendicular to the harbor main entrance, introducing wave agitation inside the harbor. Wave height is reduced close to the coastline. Computed significant wave height at the harbor entrance is about 0.3 m. Wave diffraction induced by the breakwaters will introduces additional wave height reduction resulting in a limited wave height at the quays and docks.

Figure 5. Wave height and direction for wave propagation calculated using the phase-averaged model SWAN at the Port of Almeria. Results are shown for mesh C and waves arriving from the southwest.

For waves propagating from the west, the waves arrive to the port impinging normally on the outer breakwater with a very limited wave height variation along the coast. The characteristic significant wave height is about 1 m. Due to wave diffraction the wave inside the harbor is considerably reduced. However, this modeling does not provide information to solve the waves’ phase and therefore, no detailed information on several physical magnitudes of great importance to analyze harbor operations and structures’ stability.

4. Phase-Resolving Models

4.1. Introduction

In principle, water wave motions can be modeled by the Navier-Stokes equations for
incompressible Newtonian fluids, which represent the conservation of mass and momentum. Free surface boundary conditions, ensuring the continuity of stress tensor across the free surface and the free surface is a material surface, are necessary in determining the free surface location. Both the Navier-Stokes equations and the free surface boundary conditions are nonlinear. Consequently, even when the viscous and turbulence effects can be ignored, the computational effort required for solving a truly three-dimensional wave propagation problem, which has a horizontal length scale of over hundreds or more wavelengths, is too large to be employed in real applications at this time. Consequently, wave theories have been evolving in parallel with computer power. The first wave models tried to simplify the general formulation of the problem by introducing certain assumptions that made the problem solvable.

4.2. Governing Equations for Water Waves

Consider the Cartesian coordinate system as shown in Figure 6. The free surface is given by \( z = \eta(x, y, t) \) and the rigid, impermeable bottom is given by \( z = -h(x, y) \). Assuming the water to be incompressible and a non-viscous flow, the governing equations are the continuity equation

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{at} \quad -h(x, y) \leq z \leq \eta(x, y, t)
\]

(2)

the equations of motion are

\[
\frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad \text{at} \quad -h(x, y) \leq z \leq \eta(x, y, t)
\]

\[
\frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y}
\]

\[
\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g
\]

(3)

which are known as the Euler equations.

To solve the wave problem we need to solve the continuity and momentum equations above for specific boundary conditions. These boundary conditions relate the fluid flow and the boundaries (kinematic boundary conditions) in this case applying at the free surface, bottom and lateral boundaries; but also they have to relate the boundaries with the forces acting on the fluid (dynamic boundary conditions). Being the bottom an impermeable rigid boundary, the dynamic boundary condition only applies at the free surface.

The kinematic boundary condition states that a particle at the boundary does not leave the boundary which can be expressed as

\[
w = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} \quad \text{at} \quad z = \eta(x, y, t)
\]

(4)
at the free surface and

\[ w = -u \frac{\partial h}{\partial x} - v \frac{\partial h}{\partial y} \quad \text{at } z = -h \]  

(5)

at the impermeable bottom

\[ \eta \eta = \text{at } \eta(x, y, t) \]  

(6)

where \( p_a \) is the atmospheric pressure and \( T[\eta] \) is the surface tension effect that can be expressed in terms of the first and second derivatives of the free surface.

In order to solve this set of equations initial conditions have to be added. For a non viscous flow, if the rotation is zero (zero vorticity) initially it remains zero and therefore the flow can be considered to be irrotational. Due to irrotationality a scalar velocity potential \( \phi(x, y, z, t) \) may be introduced, which is defined such that

\[ u = \frac{\partial \phi}{\partial x}; v = \frac{\partial \phi}{\partial y}; w = \frac{\partial \phi}{\partial z} \]  

(7)
The continuity equation and the relation between velocity field and velocity potential can be combined to yield Laplace’s equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$  \hspace{1cm} (8)

Combining the Euler equations an unsteady Bernoulli equation can be obtained under the irrotational ideal flow hypothesis such that

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} + gz = f(t)$$  \hspace{1cm} (9)

This equation is valid in the region \(-h(x,y) \leq z \leq \eta(x,y,t)\) and the integration constant \(f(t)\) can be eliminated by redefining the velocity potential by

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \phi}{\partial t} - f(t)$$  \hspace{1cm} (10)

By combining the Bernoulli equation (9), (10) and (6) a new dynamic free surface boundary condition can be obtained, including the effects of surface tension and atmospheric pressure

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 + \frac{p_a}{\rho} - \frac{1}{\rho} T[\eta] + g\eta = 0 \quad \text{at} \quad z = \eta(x,y,t) \quad (12a)$$

By writing the kinematic boundary conditions (4) and (5) in terms of the velocity potential the governing equations for irrotational wave motion are given by the Laplace equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{at} \quad -h(x,y) \leq z \leq \eta(x,y,t)$$  \hspace{1cm} (11)

and the three following boundary conditions

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 + \frac{p_a}{\rho} - \frac{1}{\rho} T[\eta] + g\eta = 0 \quad \text{at} \quad z = \eta(x,y,t) \quad (12a)$$

$$\frac{\partial \eta}{\partial t} + \nabla \Phi \cdot \nabla \eta = \frac{\partial \Phi}{\partial z} \quad \text{at} \quad z = \eta(x,y,t)$$  \hspace{1cm} (12b)

$$\frac{\partial \Phi}{\partial t} + \nabla \Phi \cdot \nabla h = 0 \quad \text{at} \quad z = -h(x,y)$$  \hspace{1cm} (12b)
where $\mathbf{V} = (\partial / \partial x, \partial / \partial y)$ is a two dimensional vector.

It is customary to express both free surface boundary conditions in terms of a combined free surface condition in $\Phi$ so that the free surface displacement $\eta$ remains only in the definition of the unknown domain where the problem has to be solved. Furthermore, considering the atmospheric pressure to be zero, which means that the pressure in the fluid to be determined is the pressure in excess of the atmospheric pressure and neglecting the surface tension effects the equations can be written as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{at} \quad -h(x, y) \leq z \leq \eta(x, y, t)$$  

(13a)

$$\frac{\partial \Phi}{\partial z} + \nabla \Phi \cdot \nabla h = 0 \quad \text{at} \quad z = -h(x, y)$$  

(13b)

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} + \left[ \frac{\partial}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla + \frac{1}{2} \frac{\partial \Phi}{\partial z} \frac{\partial \Phi}{\partial z} \right] \left\{ \frac{(\partial \Phi)^2}{(\partial z)^2} + |\nabla \Phi|^2 \right\} = 0 \quad \text{at} \quad z = \eta(x, y, t)$$  

(14)

Using the appropriate lateral boundary conditions, once the solution of $\Phi$ has been determined, the free surface elevation can be evaluated using the dynamic free surface boundary condition

$$\eta = \frac{1}{g} \left[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left( \frac{(\partial \Phi)^2}{(\partial x)^2} + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \right) \right] \quad \text{at} \quad z = \eta(x, y, t)$$  

(15)

Once $\eta(x, y, t)$ and $\Phi(x, y, z, t)$ has been solved, the velocities follow from the definition of the velocity potential and the expression for pressure can be obtained from the unsteady Bernoulli equation.

The governing equations presented here cannot be solved exactly, the principal difficulty being the fact that one of the unknowns in the problem, i.e., the free surface, $\eta$, is needed to define the domain where the equations have to be solved. The only way to solve this problem is by deriving approximate models. The selection of the right approximate model depends on the space and time scales of the problem considered and has to be determined depending on the values of non-dimensional parameters such as the relative depth given by $(h / L)$ or $(kh)$ and the relative wave height $(H / h)$ or wave steepness $(H / L)$, where $h$ stands for water depth, $H$ is the wave height, $L$ is the wavelength and $k = 2\pi / L$ the wave number.

4.3. Linear Wave Theory

The simplest approximate model is the linear wave theory also known as Airy or
Stokes-order one theory. In this theory linearization is performed by ignoring terms of quadratic and higher order by assuming that wave height \((H/h)\) and wave steepness \((H/L)\) are small or in terms of the Ursell parameter, if \(U_r = \left(\frac{A}{kh}\right)^2 \ll 1\), \(A \approx H/2\) being the wave amplitude. This assumption is extremely convenient because nonlinear terms are neglected and the boundary conditions referred to a known geometric surface, so that the domain is perfectly defined a priori. This approach has been proven to be very successful in many situations and the assumption of linearity permits the superposition of solutions and therefore the definition of complex motions also present in nature.

To simplify the solution of the problem in the following linear wave theory is considered for two-dimensional waves propagating on a horizontal bottom.

Considering Eqs. (11), (12a) and (12b), neglecting surface tension, the second horizontal dimension and assuming the atmospheric pressure to be zero the following equations are obtained

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{at} \quad -h \leq z \leq \eta(x,t)
\]

and the three following boundary conditions

\[
\begin{align*}
\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \right] + g\eta &= 0 \quad \text{at} \quad z = \eta(x,t) \\
\frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial x} \frac{\partial \eta}{\partial x} &= \frac{\partial \Phi}{\partial z} \quad \text{at} \quad z = \eta(x,t) \\
\frac{\partial \Phi}{\partial z} &= 0 \quad \text{at} \quad z = -h
\end{align*}
\]

Please note that Eq. (13b), known as the kinematic bottom boundary condition, has been simplified by assuming constant water depth.

For this set of equations we still have the problem of the presence of nonlinear terms and the domain defined in terms of one of the unknowns. By taking Taylor expansions around a fixed level, usually \(z = 0\), which is known, and dropping all the terms of quadratic or higher order the problem is reduced to

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{at} \quad -h \leq z \leq 0 \quad (16)
\]

and the three following boundary conditions
\[ \frac{\partial \Phi}{\partial t} + g \eta = 0 \quad \text{at } z = 0 \]  \hspace{1cm} (17)

\[ \frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z} \quad \text{at } z = 0 \]  \hspace{1cm} (18a,b)

\[ \frac{\partial \Phi}{\partial z} = 0 \quad \text{at } z = -h \]

Adding two additional conditions, i.e. the solution to be periodic in space and time

\[ \Phi(x + L, z, t) = \Phi(x, z, t) \]
\[ \Phi(x, z, t + T) = \Phi(x, z, t) \]  \hspace{1cm} (19a,b)

where \( L \) is the wave length and \( T \) the wave period.

The Laplace’s equation can be solved by using the method of separation of variables and introducing a separation variable \( k \) which can be identified with the wave number such that \( \frac{2}{\omega} \pi = \frac{2}{kL} \).

By applying the boundary conditions the solution for the velocity potential corresponding to a progressive wave is given by

\[ \Phi(x, z, t) = \frac{H}{2} g \frac{\cosh k(h + z)}{\omega \cosh kh} \sin(kx - \omega t) \]  \hspace{1cm} (20)

where \( \omega = \frac{2\pi}{T} \) is the wave frequency. Note that the vertical motion is separated from the motion in horizontal space which in this case is given by a periodic function. The argument of the periodic function \( (kx - \omega t) \) is the so-called phase of the wave.

From the dynamic boundary condition (17) it follows that the free surface elevation is expressed as

\[ \eta(x, t) = \frac{H}{2} \cos(kx - \omega t) \]  \hspace{1cm} (21)

Values of the phase which correspond to cosine values 1 or -1, represent wave crests and troughs, respectively.

In order to close the solution a dispersion relation is derived from the kinematic free surface boundary condition giving the following equation which relates wave frequency, \( \omega \), or wave period, \( T \) water depth, \( h \) and wave number or wave length, \( L \), as

\[ \omega^2 = gk \tanh kh \]  \hspace{1cm} (22)

This equation can be re-written in terms of the wave propagation speed or wave celerity,
This solution can be easily extended to two horizontal dimensions for the case of plane waves by expressing the scalar wave number as a wavenumber vector and defining the wave phase as \((\vec{k} \cdot \vec{x} - \omega t)\) such that

\[
\Phi(x, y, z, t) = \frac{H}{2} \frac{g}{\omega} \cosh k (h + z) \sin(\vec{k} \cdot \vec{x} - \omega t) \tag{24}
\]

\[
\eta(x, y, t) = \frac{H}{2} \cos(\vec{k} \cdot \vec{x} - \omega t) \tag{25}
\]

with \(\vec{k} = (k_x, k_y) = (k \cos \theta, k \sin \theta)\) the wavenumber vector and \(\theta\) the wave angle of incidence. In the dispersion equation, now \(k \equiv |\vec{k}|\).

By using the relations (7) and (24) the velocity field can be obtained

\[
u = \frac{\partial \Phi}{\partial x} = \frac{H}{2} \frac{g k_x}{\omega} \cosh k (h + z) \cos(\vec{k} \cdot \vec{x} - \omega t)
\]

\[
v = \frac{\partial \Phi}{\partial y} = \frac{H}{2} \frac{g k_y}{\omega} \cosh k (h + z) \cos(\vec{k} \cdot \vec{x} - \omega t)
\]

\[
w = \frac{\partial \Phi}{\partial z} = \frac{H}{2} \frac{g k}{\omega} \sinh k (h + z) \sin(\vec{k} \cdot \vec{x} - \omega t)
\]

and the pressure follows from Bernoulli equations by dropping the nonlinear terms

\[
p(x, y, z, t) = -\rho g z - \rho \frac{\partial \Phi}{\partial t} = -\rho g z - \rho \cosh k (h + z) \sinh k h \eta(x, y, t)
\]

The pressure has two different contributions, the hydrostatic component and the dynamic component which is proportional to the free surface displacement.

For further applications, it is necessary to explain that the energy associated to the wave travels at the group celerity, \(C_g\), which is related to the wave celerity by the following relation

\[
C_g = \frac{1}{2} \left[ 1 + \frac{kh}{2 \sinh 2kh} \right] C
\]

Note that in the two horizontal dimensions space, both the celerity and group celerity...
are vectors in the direction of wave propagation. The linear solution presented provides an excellent means to find new solutions based on the principles of superposition representing more realistic wave free surface such as standing or quasi-standing waves (resulting from the interaction of incident and reflected waves); short-crested waves (resulting from the interaction of wave propagating in different directions) or wave groups (resulting from wave propagating in the same direction with slightly different frequencies). Moreover, the superposition of a large number of wave components with different wave amplitudes, frequencies and directions may be a good approximation to represent a real wave spectrum as the ones calculated using phase-averaged models.

However, at this point, the solution presented has been found assuming a horizontal bottom, an important shortcoming for real applications.

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bottom friction is modeled with a term quadratic in the horizontal fluid velocity]

Biographical Sketches

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