SIGNAL THEORY

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Contents

Summary

As the sole means for communication and for perception of the world, signals play an integral role in life. The light signals falling on our retinas and the air pressure signals causing vibrations of the inner bone structure of the ear allow us to see and hear the world around us. Our vocal tracts allow us to form air pressure signals for communicating with others. Devices have been developed to transmit and receive signals over vast distances in order to extend our ability to communicate. Our ability to perceive the world has similarly been enhanced by building devices that form signals describing the fine structure of materials or the characteristics of distant objects. This chapter reviews the theory relating to signals and the systems through which they pass. After describing some of the properties of various classes of signals and systems, the important operations of convolution and Fourier analysis will be described. Convolution describes a common class of system – the linear, time-invariant system and Fourier analysis provides a powerful technique for analyzing and processing signals and linear time-invariant systems. Most modern signal processing is performed by digital computer, which requires continuous time signals to be sampled and quantized. The implications of this process are discussed. Finally, the theory relating to random signals (where the characteristics of the signals can only be described in a probabilistic manner) is reviewed.

1. Introduction

A signal describes information. The information can include a wide variety of physical phenomena, for example a sound wave or a pattern of light on a surface. The signal conveys this information in the form of fluctuations in a function of one or more independent variables. In the case of a sound wave, the fluctuations in air pressure with time at a particular point in space can be a considered a signal. In the case of a two dimensional, monochromatic image, the light intensity as a function of two spatial coordinates is a signal.

Often it is desirable to modify the form in which the signal represents information. Any such process that transforms a signal according to some specified rule is referred to as a system. This might be for purposes of instrumentation, where a physical signal is transformed into an electrical signal using a transducer; for analysis/signal processing, where the independent variable(s) might be transformed into another "domain" prior to further analysis/processing; communications where a carrier signal is modulated by a message signal; or a host of other purposes. Often the signal is sampled and quantized to produce a discrete-time, discrete-valued (digital) signal, which allows for sophisticated analysis and signal processing.

This chapter will discuss the theory of signals in one dimension (*i.e.* signals with only one independent variable.) The concepts easily extend to signals with more than one independent variable. For the purposes of this chapter, the independent variable will be assumed to be time, as that is the most common variable for a one dimensional signal. The time variable will be mathematically represented by the real number t for continuous time signals, or by the integer n in discrete time signals.

2. Properties of Signals

A signal can be classified as periodic or aperiodic; discrete or continuous time; discrete of continuous valued; or as a power or energy signal. The following defines each of these terms. In addition, the signal-to-noise ratio of a signal corrupted by noise is defined.

2.1 Periodic / Aperiodic

A periodic signal repeats itself at regular intervals. In general, any signal $x(t)$ for which

$$x(t) = x(t - T) \tag{1}$$

for all t is said to be *periodic*. The fundamental period of the signal is the minimum positive, non-zero value of T for which Eq. (1) is satisfied. If a signal is not periodic, then it is *aperiodic*.

2.2. Symmetric / Asymmetric

There are two types of signal symmetry: odd and even. A signal $x(t)$ has *odd symmetry* if and only if

$$x(-t) = -x(t) \tag{2}$$

for all t. It has *even symmetry* if and only if

$$x(-t) = x(t). \tag{3}$$

2.3. Discrete/Continuous Time

A continuous time signal is defined for all values of t. A discrete time signal is only defined for discrete values of $t = ..., t_{-1}, t_0, t_1, ..., t_n, t_{n+1}, t_{n+2}, ...$ It is uncommon for the interval between t_n and t_{n+1} to change with n. The interval is most often some constant value referred to as the sampling time,

$$T_s = t_{n+1} - t_n. \tag{4}$$

It is often convenient to express discrete time signals as

$$x(nT_s) = x[n]. \qquad (5)$$

That is, if $x(t)$ is a continuous-time signal, then $x[n]$ can be considered as the n^{th} sample of $x(t)$.

Sampling of a continuous-time signal $x(t)$ to yield the discrete-time signal $x[n]$ is an important step in the process of digitizing a signal. This will be discussed further in Section 6.

2.4. Discrete/Continuous Valued

A continuous valued signal can take on any real (or complex) value. Examples include the water level in a water tank or the speed of a vehicle. Due to quantization, a discrete valued signal can only take on only a predefined subset of real (or complex) values. An example is the output of an analogue to digital converter (*i.e.* a continuous-time, continuous valued signal that has been digitized.) Such a signal consists of integers only.

Note that although a signal might be discrete-valued, it is not necessarily a discrete-time time signal.

2.5. Signal Energy and Power

When the strength of a signal is measured, it is usually the signal power or signal energy that is of interest. The signal power of $x(t)$ is defined as

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt \qquad (6)$$

and the signal energy as

$$E_x = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\,dt \qquad (7)$$

A signal for which P_x is finite and non-zero is known as a *power signal*. A signal for which E_x is finite and non-zero is known as an *energy signal*. P_x is also known as the *mean-square* value of the signal.

Signal power is often expressed in the units of decibels (dB). The decibel is defined as

$$P_x \text{dB} = 10\log\left(\frac{P_x}{P_0}\right) \qquad (8)$$

where P_0 is a reference power level, usually equal to one squared SI unit of the signal.

For example if the signal is a voltage then the P_0 is equal to one square Volt.

As an example, the sinusoidal test signal of amplitude A

$$x(t) = A\sin(\omega t) \tag{9}$$

has energy E_x that tends to infinity and power $P_x = \frac{1}{2}A^2$, or in decibels $20\log(A) - 3$ dB. The signal is thus a power signal.

2.6 Signal to Noise Ratio

Any measurement of a signal necessarily contains some random noise in addition to the signal. In the case of additive noise, the measurement is

$$x(t) = x_\text{s}(t) + x_\text{n}(t) \tag{10}$$

where $x_\text{s}(t)$ is the signal component and $x_\text{n}(t)$ is the noise component.

The signal to noise ratio is defined as

$$SNR_x = \frac{P_\text{s}}{P_\text{n}} \tag{11}$$

or in decibels,

$$SNR_x = 10\log\left(\frac{P_\text{s}}{P_\text{n}}\right)\text{dB} \tag{12}$$

The signal to noise ratio is an indication of how much noise is contained in a measurement.

3. Elementary Signals

There are various classes of functions that are useful for analysis of signals. These are: the unit impulse; the unit step; and complex exponentials.

3.1 Unit Impulse

The unit impulse, $\delta(t)$, is a function that is zero for all $t \neq 0$ and for which

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{13}$$

One way to define the unit impulse function is

$$\delta(t) = \lim_{\Delta \to \infty} \left[\delta_\Delta(t) \right] \quad (14)$$

where

$$\delta_\Delta(t) = \begin{cases} 1/\Delta & 0 < t \le \Delta \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

Figure 1 shows how $\delta(t)$ is plotted.

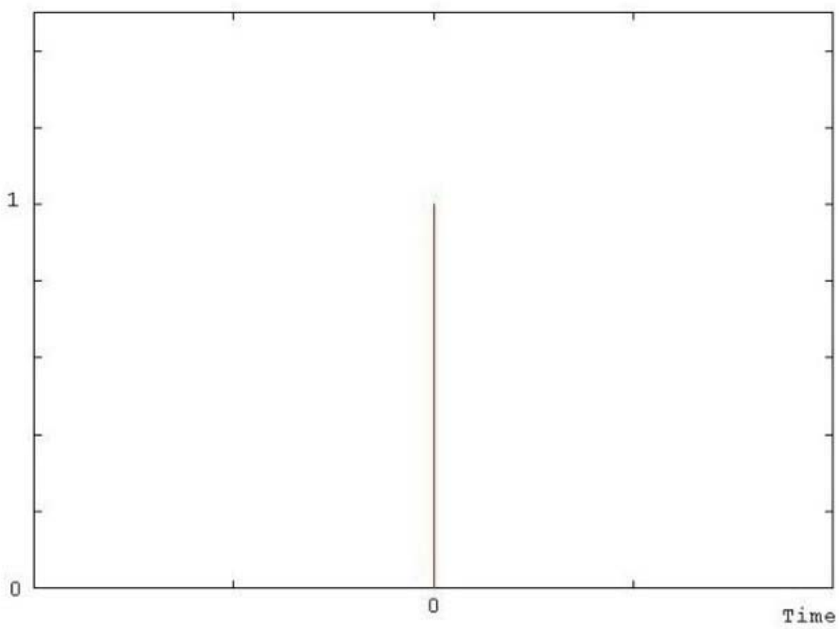

Figure 1: The unit impulse function

3.2. Unit Step

The unit step function is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases} \quad (16)$$

and it is discontinuous at $t = 0$.

The unit step and the impulse functions are related to each other by

$$u(t) = \int_{-\infty}^{t} \delta(t)dt \quad (17)$$

$$\delta(t) = \frac{du(t)}{dt} \quad (18)$$

Figure 2 shows a plot of $u(t)$.

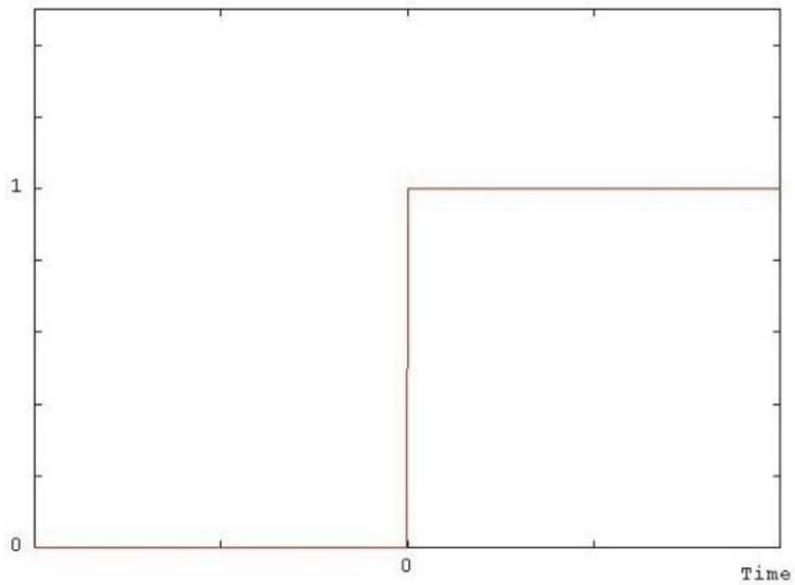

Figure 2: The unit step function

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Biographical Sketch

Alexander Lowe graduated from Curtin University of Technology in Perth, Australia with a B.Eng.

(electronics and communications) degree in 1988 and a Ph.D. (corrosion instrumentation) degree in 2002. His Ph.D. project involved the measurement and analysis of signals in corrosion systems. As a result of the research, he developed a set of signal processing techniques designed for corrosion evaluation. In 2002, Dr. Lowe gave lectures on Digital Communications and performed research on improving the life of lead-acid batteries. He has designed and manufactured various instruments for use in corrosion science research.