POPOV AND CIRCLE CRITERION

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Keywords: Absolute stability, Bounded real transfer matrix, Circle criterion, Kalman-Yakubovich lemma, Popov criterion, Positive real transfer matrix, Sector condition.

Contents

Summary

Control systems are usually described by models which are not precise and there is an amount of uncertainty. In particular, imprecise nonlinear funtions can be characterized by functions having upper and lower bounds defining a certain sector in the state space. An equilibrium point $\mathbf{x} = \mathbf{0}$ of a dynamical system, which is globally asymptotically stable for all nonlinear functions located in this sector, is called absolutely stable. The system is robustly stable against uncertainties. The article lists a number of sufficient conditions for absolute stability, among others the well-known circle and Popov criteria.

1. Introduction

This chapter on the Popov and circle theorems for the problem of absolute stability is based on *Stability Theory* and uses the same notation. The discussion of the stability problem in *Stability Theory* was based on the assumption that the dynamical system (1) or (2)-(3) (in *Stability Theory*) is described exactly. But this is not always true. Particularly, the exact description of nonlinear characteristics is difficult. Therefore, investigations have been carried out to overcome these uncertainty problems and to guarantee robust stability of a dynamical system with respect to model uncertainties, particularly with respect to unknown nonlinear characteristics. In this connection a linear time-invariant nominal system is assumed which is asymptotically stable. The model uncertainties have to satisfy certain restrictions. Then the question arises if the real system remains stable for this class of uncertainties or if it becomes unstable.

The starting point of these considerations is with the linear time-invariant system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\,, \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{2}$$

describing a nominal behavior. The input vector $\mathbf{u}$ represents inaccurately modeled effects or nonlinear actuator characteristics which depend on certain output variables (2):

$$\mathbf{u} = -\mathbf{k}(\mathbf{y},t),\ \mathbf{k}(\mathbf{0},t) = \mathbf{0}. \qquad (3)$$

The vector function **k** may be very general; it is restricted only by a sector condition

$$\left[\mathbf{k}(\mathbf{y},t) - \mathbf{K}_1\mathbf{y}\right]^{\mathrm{T}}\left[\mathbf{k}(\mathbf{y},t) - \mathbf{K}_2\mathbf{y}\right] \le 0 \qquad (4)$$

for certain matrix bounds $\mathbf{K}_1$, $\mathbf{K}_2$. If the dimensions of **u** and **y** are equal, $r = m$, and the feedback (3) decomposes into *m* independent channels,

$$u_i = -k_i(y_i,t),\ i = 1,\ldots,m, \qquad (5)$$

then the matrices $\mathbf{K}_1$, $\mathbf{K}_2$ are diagonal, $\mathbf{K}_1 = \mathrm{diag}(k_{1i})$ $\mathbf{K}_2 = \mathrm{diag}(k_{2i})$, and (4) reduces to the usual sector condition

$$k_{1i} \le \frac{k_i(y_i,t)}{y_i} \le k_{2i},\ i = 1,\ldots,m\ \ (k_{1i} < k_{2i}, i = 1,\ldots,m). \qquad (6)$$

We assume that the system (1)-(2) is completely controllable and completely observable (otherwise the problem decomposes into trivial or smaller problems of systems of lower order) and asymptotically stable for $\mathbf{u} = \mathbf{0}$. Now the problem of absolute stability can be defined. It is also known as the Lur'e (or Lur'e-Postnikov) problem.

Definition 1 (Absolute Stability): The completely controllable and completely observable system (1)-(2), which is asymptotically stable for $\mathbf{u} = \mathbf{0}$, is absolutely stable if for all uncertain output feedbacks (3) satisfying the sector condition (4) [or (6)] the equilibrium point $\mathbf{x} = \mathbf{0}$ is globally asymptotically stable.

Here, global asymptotic stability means that the equilibrium point is asymptotically stable and its domain of attraction is the whole space, i. e. all solutions approach the equilibrium point.

Absolute stability guarantees asymptotic stability in the whole for the complete class of nonlinearities (3) satisfying the sector condition (4) [or (6)]. Therefore, the stability results of Section 3 are valid not only for uncertain nonlinearities but also for well defined nonlinearities within the allowed sector.

Sometimes the sector condition is given for $\mathbf{K}_1 = \mathbf{0}$ and $\mathbf{K}_2 = \mathbf{K}$,

$$\mathbf{k}^{\mathrm{T}}(\mathbf{y},t)\left[\mathbf{k}(\mathbf{y},t) - \mathbf{K}\mathbf{y}\right] \le 0 \qquad (7)$$

or

$$0 \le \frac{k_i(y_i,t)}{y_i} \le k_i\,,\ i = 1,\ldots,m\,, \tag{8}$$

instead of (4) or (6), respectively. But the sector condition (7) or (8) is obtained by

$$\mathbf{u} = -\mathbf{K}_1\mathbf{y}\,,\ \mathbf{K} = \mathbf{K}_2 - \mathbf{K}_1 \tag{9}$$

from (4) and (6) for a modified system. Therefore, both formulations are equivalent.

The requirement of the asymptotic stability of the system (1) for $\mathbf{u} = \mathbf{0}$ seems to be a strong restriction for the dynamic systems under consideration, but this is not the case. In a two step procedure

$$\mathbf{u}(t) = \mathbf{u}_1(t) + \mathbf{u}_2(t) \tag{10}$$

in the first step, $\mathbf{u}_1(t)$ will be designed by methods of linear control theory to stabilize the linear system, and in the second step $\mathbf{u}_2(t)$ will be considered in the sense of the absolute stability problem. Typically, the problem of absolute stability is the demand for robust stability of closed-loop control systems which are described and designed approximately by linear methods but which additionally are influenced by uncertain nonlinear effects.

2. Kalman-Yakubovich-Lemma

In Section 3 sufficient criteria for absolute stability are presented. The related proofs use suitably chosen Lyapunov functions where special relations between these functions and the transfer function matrix

$$\mathbf{G}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D} \tag{11}$$

hold. Some properties of (11) and some related lemmas are summarized in the following.

Definition 2 (Bounded Real Transfer Matrix): The transfer function matrix (11) is strictly bounded real if there exists a number γ such that

$$\mathbf{G}(s) \text{ is asymptotically stable,} \tag{12}$$

$$\mathbf{G}^*(j\omega)\mathbf{G}(j\omega) < \gamma^2\mathbf{I}_r \text{ for all real } \omega \tag{13}$$

$$\mathbf{D}^{\mathrm{T}}\mathbf{D} < \gamma^2\mathbf{I}_r\,. \tag{14}$$

Condition (12) means asymptotic stability of the linear system for $\mathbf{u} = \mathbf{0}$. In (13) $\mathbf{G}^*(j\omega) = \mathbf{G}^{\mathrm{T}}(-j\omega)$.

Lemma 1 (Bounded Realness): The transfer function matrix (11) of a completely controllable and observable system (1)-(2) is strictly bounded real if and only if there exist matrices $\mathbf{P} = \mathbf{P}^{\mathrm{T}} > \mathbf{0}$, $\mathbf{L}$ and $\mathbf{W}$ such that

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{C}^{\mathrm{T}}\mathbf{C} + \mathbf{L}^{\mathrm{T}}\mathbf{L} = \mathbf{0}, \tag{15}$$

$$\mathbf{P}\mathbf{B} + \mathbf{C}^{\mathrm{T}}\mathbf{D} + \mathbf{L}^{\mathrm{T}}\mathbf{W} = \mathbf{0}, \tag{16}$$

$$\gamma^2\mathbf{I}_r - \mathbf{D}^{\mathrm{T}}\mathbf{D} = \mathbf{W}^{\mathrm{T}}\mathbf{W} > \mathbf{0}. \tag{17}$$

Asymptotic stability (12) follows immediately from (15) and vice versa. The requirement (14) follows by (17). Finally, the relation

$$\gamma^2\mathbf{I}_r - \mathbf{G}^*(j\omega)\mathbf{G}(j\omega) = \mathbf{H}^*(j\omega)\mathbf{H}(j\omega) > \mathbf{0} \tag{18}$$

can be derived from (15) – (17) where

$$\mathbf{H}(j\omega) = \mathbf{L}(j\omega\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} + \mathbf{W} \tag{19}$$

is defined. Therefore, (13) is satisfied. The reversal statement is obtained by spectral factorization.

Definition 3 (Positive Real Transfer Matrix): A square $m \times m$ transfer matrix (11) is strictly positive real if

$$\mathbf{G}(s) \text{ is asymptotically stable,} \tag{20}$$

$$\mathbf{G}^*(j\omega) + \mathbf{G}(j\omega) > 0 \text{ for all real } \omega \text{, and} \tag{21}$$

$$\mathbf{D}^{\mathrm{T}} + \mathbf{D} > \mathbf{0}. \tag{22}$$

Lemma 2 (Positive Real Transfer Matrices; Kalman-Yakubovich): A square transfer matrix (11) of a completely controllable and observable system (1)-(2) is strictly positive real if there exist matrices $\mathbf{P} = \mathbf{P}^{\mathrm{T}} > \mathbf{0}$, $\mathbf{L}$ and $\mathbf{W}$ such that

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{L}^{\mathrm{T}}\mathbf{L} = \mathbf{0}, \tag{23}$$

$$\mathbf{P}\mathbf{B} - \mathbf{C}^{\mathrm{T}} + \mathbf{L}^{\mathrm{T}}\mathbf{W} = \mathbf{0}, \text{ and} \tag{24}$$

$$\mathbf{D}^{\mathrm{T}} + \mathbf{D} = \mathbf{W}^{\mathrm{T}}\mathbf{W} > \mathbf{0}. \tag{25}$$

The proof proceeds analogously to the preceding one. Stability is assured by (23). Inequality (22) follows immediately from (25). Finally (21) is shown by (23 – 25) resulting in

$$\mathbf{G}^*(j\omega)+\mathbf{G}(j\omega)=\mathbf{H}^*(j\omega)\mathbf{H}(j\omega)>\mathbf{0} \quad (26)$$

where $\mathbf{H}(j\omega)$ has been defined by (19).

With the aid of Lemmas 1 and 2 some sufficient criteria for absolute stability can be proven.

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Biographical Sketch

Peter C. Müller

1940	Born at Stuttgart, Germany
1965	Degree of "Diplom-Mathematiker" from the Technical University of Stuttgart
1966 - 1981	Institute of Mechanics, Technical University of München
1970	Dr. rer. nat. ("Time-optimal alignment of inertial platforms")
1974	Habilitation in Mechanics ("Stability and matrices")
1978	Professor of Mechanics
Since 1981	University of Wuppertal Professor of Safety Control Engineering
1983 - 1985	Dean of the Department of Safety Engineering
Since 1987	Head of the Institute of Robotics
1995 – 1999	Vice Rector of the University of Wuppertal
Member of	GAMM, VDI, IEEE
Co-editor of	"Archive of Applied Mechanics" " Mathematical and Computer Modelling of Dynamical Systems" "ZAMM"

1988 - 2000	Selection Committee of the Alexander von Humboldt Foundation
1994 - 2000	Committee for Postgraduate Research Groups of the German Research Council (DFG)
Research:	Control of Mechanical Systems Nonlinear Control Systems Descriptor Systems Robotics and Mechatronics Fault Detection and Supervisory Control
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