FAULT DIAGNOSIS FOR NONLINEAR SYSTEMS

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Keywords: fault diagnosis, fault detection and isolation, nonlinear system, bilinear system, state affine system, control affine system, Takagi-Sugeno fuzzy system, differential algebraic equations, residual generator, state observer, high gain observer, analytical redundancy relations.

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Summary

The principle of the design of residual generators for systems described by bilinear models, by state affine or control affine state equations, by Takagi-Sugeno fuzzy model and by differential algebraic equations is presented. Observer based methods are considered for the first four classes of systems, and analytical redundancy relations are introduced for the last class of models.

1. Introduction

The performance of model based fault detection and isolation (FDI) systems depends heavily on the quality of the available model for the supervised process. As most physical systems are inherently nonlinear, precise models must take this nonlinearity into account, in order to achieve reliable FDI over the whole working range of the supervised process. The tools to be used for the design of FDI systems depend on the considered class of models. A distinction can be made between situations where the
physical laws governing the process allow one to deduce a model with a specific structure, and in the case where black box models are used. The latter are identified from experimental data without using physical knowledge. The first class of models will be considered here, except for Takagi-Sugeno fuzzy models which might be identified from experimental data without using physical relations or might be deduced from a smooth nonlinear model based on the laws of physics. Other types of black box models like neural nets are not considered; the reader is referred to the bibliography for that topic.

A typical FDI system is made of two parts: a residual generator and a decision system. The first generates signals called residuals that are nominally equal to zero in the absence of faults (after possible vanishing of a transient due to initial conditions), and some of the residuals become distinguishably different from zero upon occurrence of a fault. The decision system analyzes the pattern of zero and non zero residuals in order to determine the most likely faulty component(s). It typically relies on statistical methods for change detection and isolation and/or on methods based on qualitative models. As these topics are addressed in other articles, the emphasis here will be on residual generation.

The simplest problem of residual generation, namely the design of residuals aimed at detecting and isolating two faults will be considered, in order to illustrate the principle behind the design methods without introducing cumbersome notations. The solution of this problem, called the fundamental problem of residual generation (FPRG), will be discussed successively for systems described by bilinear, state affine and control affine state equations, as well as polynomial differential algebraic equations (DAE’s). Finally the design of a fault detection system based on Takagi-Sugeno fuzzy model will be described. For the latter class of systems, the issue of fault isolation is not considered.

2. Model Classes

The following notations are used in all the models. \( x \) denotes the \( n \)-dimensional state vector, \( u \), the \( m \)-dimensional vector of known inputs, \( y \) the \( p \)-dimensional vector of output measurements. The two possible faults to be considered are represented by the variables \( v_1 \) and \( v_2 \). These are scalar functions of time that have zero value in the absence of fault. Upon occurrence of fault 1, \( v_1 \) takes arbitrary unknown nonzero values, while \( v_2 \) remains equal to zero, and reciprocally for fault 2.

Bilinear and state affine systems are described by:

\[
\begin{align*}
\dot{x} &= A(u)x + Bu + e_1(x)v_1 + e_2(x)v_2 \\
y &= Cx
\end{align*}
\]  

(1)

where \( \dot{x} \) denotes the derivative of \( x \) with respect to \( t \). In the case of a bilinear system

\[ A(u) = A_0 + \sum_{i=0}^{m} u_i A_i, \]

(2)

where \( A_i, i = 0, ..., m \) are \( n \times n \) constant matrices and \( u_i \) denotes the \( i^{th} \) component of vector \( u \), while for state affine systems, \( A(u) \) is supposed to depend smoothly on the entries of \( u \). In both cases, \( e_i(x), i = 1, 2, \) are smooth functions of the entries of \( x \). Note that the way the system is called results from the form of the equations when \( v_1 = v_2 = 0 \).
Control affine systems are of the following form:

\[
\begin{align*}
\dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i + e_1(x)v_1 + e_2(x)v_2 \\
y &= h(x)
\end{align*}
\]  

(3)

where \(f, g_i, i = 1, \ldots, m, e_i, i = 1, 2, \) and \(h\) are smooth functions of their arguments.

Finally systems described by differential algebraic equations (DAE’s) will be considered:

\[
f_i\left(x, \overline{u}^{s_u}, \overline{y}^{s_y}, \overline{v}_1^{s_v}, \overline{v}_2^{s_v}\right) = 0, \quad i = 1, \ldots, n_{eq}
\]

(4)

where \(\overline{u}^{s_u}\) denotes the set made of \(u\) and all its time derivatives up to order \(s_u, u^{(s_u)}\), and a similar definition holds for \(\overline{y}^{s_y}\) and \(\overline{v}_j^{s_v}, j = 1, 2\). \(f_i\) are polynomials in \(u, y, v_j, j = 1, 2, \) and their derivatives, as well as \(x\). In the sequel the notation \(\overline{u}\) is used to denote \(u\) and its derivatives up to a finite unspecified order, and similarly for \(\overline{y}, \overline{v}_1, \overline{v}_2\).

The signals \(u, v_i, i = 1, 2\) respectively belong to the classes \(U\) and \(V_i, i = 1, 2\) of piecewise continuous functions such that, for any initial state, any trajectory of system Eqs. (1), (3), or (4) associated to those inputs is defined in the whole interval \([0, +\infty]\).

The description of the Takagi-Sugeno fuzzy model to be used is deferred to section Error! Reference source not found., as a different problem is also considered for this class of models.

Let us now state the problem of residual generator design.

3. Residual Generator Design

3.1. Problem Statement

The so-called fundamental problem of residual generation (FPRG) can be stated as follows:

**Definition 1 FPRG for system Eqs. (1), (3), or (4):** Design a filter with inputs \(u\) and \(y\) (where \(y\) is the output of the considered system Eqs. (1), (3), or (4)) of which the output vector \(r\) fulfills the following requirements.

1. There exists a set \(\overline{U}\) in \(U\) such that, in the absence of fault \(v_1\) (namely when \(v_1 = 0\)), \(r\) decays asymptotically to zero for any \(u\) in \(\overline{U}\), any \(v_2\) in \(V_2\), and any initial condition of the filter and of system Eqs. (1), (3), or (4).
2. When \(v_1 \neq 0\) for all \(t > t_0\), \(r\) is non zero for at least some \(t > t_0\).

Such a filter is called a residual generator for detection of \(v_1\). Its output is called a residual vector. It will become non zero only upon occurrence of fault 1, once the transient due to initial conditions has decayed to zero. If a second residual generator is designed by interchanging the role of \(v_1\) and \(v_2\) in the above problem statement, the
output of this filter will only be different from zero upon occurrence of fault 2. By monitoring both residual vectors it is thus possible to detect and isolate the faults; namely, to determine which fault occurred.

This idea can be generalized to situations were more than two faults can occur. Moreover, one need not always require that each residual becomes non-zero upon the occurrence of only one specific fault. Indeed, let \( v_j, j = 1, \ldots, n_f \) be the possible faults, and let \( r_k, k = 1, \ldots, n_r \) be the set of residual vectors aimed at detecting and isolating each fault. The notion of a coding set is introduced to characterize the residual vectors which are affected by a given fault. For instance, the coding set \( \Omega_j \) associated to the \( j^{th} \) fault is the set of residual vectors which becomes non-zero upon the occurrence of fault \( j \) (i.e. when \( v_j \neq 0 \)). In order to achieve fault isolation under the assumption that no simultaneous faults occur, it suffices to associate to each fault a different coding set. The particular choice where \( n_f = n_r \) and \( \Omega_j = \{ j \} \) assures that simultaneous faults can be detected and isolated.

For models Eqs. (1) and (3), one way to solve the FPRG is first to extract a subsystem of which \( v_2 \) is not an input, and for which an asymptotic observer can be designed. The second step is precisely the design of this observer. The output estimation error then fulfills requirement 1 of the FPRG, and it qualifies as a residual, provided that it also verifies condition 2. The introduction of the set \( \mathcal{U} \) in the problem statement is due to the fact that nonlinear systems might not be observable for all inputs. Convergence of an observer for a nonlinear system might thus only be achieved for signals \( u \) belonging to a subset of \( \mathcal{U} \).

For model Eq. (4), a way to solve the FPRG is to eliminate the state \( x \) and fault \( v_2 \) together with its derivatives by appropriate operations on the set of DAE’s. The resulting DAE’s only depend on \( u, \overline{v}_1 \) and \( \overline{v}_2 \). The polynomial functions that make the left hand side of those DAE’s can be evaluated for given experimental data, with \( \overline{v}_1 \) set to zero. The results will be zero in the absence of fault 1 (\( \overline{v}_1 = 0 \)) and non-zero otherwise. Hence, the corresponding vector signal qualifies as a residual.

**Remark 1** The sensitivity of the residual to fault 1 expressed in condition 2 can be translated in a mathematical framework in different ways. A mathematically tractable statement of this condition also depends on the class of models, but one will not enter into those details here.

**Remark 2** Saying that a residual asymptotically decays to zero or is equal to zero is obviously a theoretical statement. In practice, due to modeling uncertainties, model discretization to handle sampled data, approximation of derivatives of the signals (for DAE models), and measurement noise, a residual will not decay to zero or be equal to zero, but its norm will on average remain under a fixed bound. A faulty behavior will then be characterized by a norm of the residual larger than this bound.

### 3.2. Principle of the Solution of the FPRG for Bilinear Systems
As already stated, the solution relies on the determination of a system with $v_1$ as the only unknown input, from the original state space model of the plant, Eq. (1).

To this end, $(m+1)$ output injection maps

$$D_i : \mathbb{R}^p \to \mathbb{R}^n, i = 0, \ldots, m,$$

and an output mixing map

$$L : \mathbb{R}^p \to \mathbb{R}^p$$

are introduced in order to define the following system class:

$$\dot{x} = (A_i + D_0 C)x + \sum_{i=1}^{m} u_i (A_i + D_i C)x - D_1 y - \sum_{i=1}^{m} u_i D_i y + Bu + e_1(x)v_1 + e_2(x)v_2$$  \hspace{1cm} (7)

$$z = LCx$$  \hspace{1cm} (8)

This system has the same dynamics as system Eq. (1). Equation (7) can be rewritten in the following compact form, with notations similar to that of Eq. (1):

$$\dot{x} = (A(u) + D(u)C)x - D(u)y + Bu + e_1(x)v_1 + e_2(x)v_2$$  \hspace{1cm} (9)

To proceed, set $\mathcal{D} = (D_0, \ldots, D_m)$, and consider the subspace $S(L, \mathcal{D})$ spanned by the row vectors:

$$L_i, C, L_i, C(A_j + D_j C), \ldots, L_i, C \prod_{k=1}^{l} (A_{j_k} + D_{j_k} C), \ l \geq 2, j, j_k \in \{0, \ldots, m\}$$  \hspace{1cm} (10)

where $L_{i,*}$ is the $i^{th}$ row of $L$. Assume that $D_i$, $i = 0, \ldots, m$ and $L$ have been determined so that this subspace has dimension $d < n$. Let $(T_1, \ldots, T_d)$ be a basis of $S(L, \mathcal{D})$, and let $(T_{d+1}, \ldots, T_n)$ be $(n-d)$ row vectors such that the matrix

$$T = \begin{pmatrix} T_1, & \vdots & \vdots & T_n, \end{pmatrix}$$  \hspace{1cm} (11)

is invertible. Set

$$\zeta = Tx = \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}$$  \hspace{1cm} (12)

where

$$\zeta^1 = \begin{pmatrix} T_1, x \\ \vdots \\ T_d, x \end{pmatrix},$$  \hspace{1cm} (13)

$$\zeta^2 = \begin{pmatrix} T_{d+1}, x \\ \vdots \\ T_n, x \end{pmatrix}.$$  \hspace{1cm} (14)

In the new system of coordinates $(\zeta_1, \ldots, \zeta_n)$, system Eqs. (7), (8) can be written:

$$\dot{\zeta}^1 = A^1(u)\zeta^1 - D^1(u)y + B^1u + e_1^1(\zeta)v_1 + e_2^1(\zeta)v_2$$  \hspace{1cm} (15)
\[
\dot{\xi}^i = A^{ii}(u)\xi^i + A^{ij}(u)\xi^k - D^j(u)y + B^j u + e^i_1(\xi) v_1 + e^i_2(\xi) v_2
\]

\[
z = \widehat{C}\xi^i
\]

where

\[
T(A(u) + D(u)C)T^{-1} = \left(\begin{array}{cc}
A^{ii}(u) & 0 \\
A^{ij}(u) & A^{jj}(u)
\end{array}\right)
\]

(16)

\[
D^i(u)
\]

(17)

\[
D^j(u)
\]

(18)

\[
\begin{bmatrix}
e^i_1(\xi) \\
e^i_2(\xi)
\end{bmatrix} = Te_i(T^{-1}\xi), \ i = 1, 2,
\]

(19)

\[
\begin{bmatrix}
B^i \\
B^j
\end{bmatrix} = TB
\]

(20)

\[
[C \ 0] = LCT^{-1}
\]

(21)

This state transformation is a generalization to bilinear systems of the well known approach to separate a linear system into its observable and non observable parts. By construction, the bilinear system:

\[
\dot{\xi}^i = A^{ii}(u)\xi^i
\]

(22)

\[
z = \widehat{C}\xi^i
\]

(23)

is observable. Moreover, if

\[
\begin{bmatrix}
T_{1,1} \\
\vdots \\
T_{m,1}
\end{bmatrix} e_2(T^{-1}\xi) = 0
\]

(24)

for all \(\zeta \in \mathbb{R}^n\) then

\[
\overline{e}^i_2(\xi) = 0
\]

(25)

and \(v_1\) is the only unknown input of system Eqs. (15), (17). The issue of determining \(L\) and \(D_i, i = 0, \ldots, m\) in order to obtain a system of the form Eqs. (15), (17) with

\[
\overline{e}^i_2(\xi) = 0
\]

(26)

is considered below. The next step is the design of an exponential observer for this system when \(v_1 = 0\). It can be shown that, for any input signal \(u\) in a specific set \(\overline{U}\), the following system is such an observer:

\[
\begin{cases}
\dot{\zeta}^i = \overline{A^{ii}(u)}\xi^i - \overline{D^i(u)}y + \overline{B^i} u - S^{-1}\overline{C^T}\left(\overline{C}\xi^i - z\right) \\
S = -0S - \overline{A^{ii}(u)}^T S - S\overline{A^{ii}(u)} + \overline{C^T}\overline{C}
\end{cases}
\]

(27)
where $\theta \in \mathbb{R}$ is positive, and can be adjusted to obtain suitable observer dynamics. $S(0)$ should be positive definite, and $\hat{\zeta}^i(0)$ can be set equal to zero in the absence of a priori information. The set $\overline{U}$ contains the, so-called, regularly persistent inputs for system Eqs. (23), (24). Roughly speaking, for such inputs, the observability Gramian that is associated with system Eqs. (23), (24), computed over specific time intervals, is positive definite.

The output reconstruction error:
$$r = z - \overline{C}\hat{\zeta}^i = \overline{C}e$$

where
$$e = \zeta^i - \hat{\zeta}^i,$$

fulfills condition 1 of the FPRG.

Indeed, notice that $e$ is governed by:
$$\dot{e} = \left( \overline{A}T(u) - S^{-1}\overline{C}^T\overline{C} \right) e + \overline{C}^T(\zeta) v_1$$

with $\zeta$ obtained from Eqs. (15), (16). Hence, given the exponential convergence of the observer for all $u \in \overline{U}$, $r$ exponentially decays to zero when $v_1 \equiv 0$ for any $v_2 \in V_2$. To qualify as a residual it must also verify condition 2 of the FPRG.

Clearly a necessary condition to assure sensitivity to fault 1 is that
$$\overline{e}_1^i(\zeta) \neq 0,$$

which means that
$$\mathcal{E}_1 = \text{span}\{e_1(x), x \in \mathbb{R}^n\}$$

cannot be included in the kernel of
$$\begin{pmatrix} T_{i,1} \\ \vdots \\ T_{i,m} \end{pmatrix}.$$  \hspace{1cm} (34)

To be able to state a necessary and sufficient condition for the existence of a solution to the FPRG, the notion of $(C, A)$-unobservability subspace must be introduced, where $A = (A_0, \ldots, A_m)$. Such a subspace is the annihilator, $S(L, D)^\perp$, of the subspace $S(L, D)$ for given $L$ and $D$, namely the set of all $p \in \mathbb{R}^n$ such that, for any row vector $q$ in $S(L, D)$, $qp = 0$. In the solution of the FPRG, the interest is in $(C, A)$-unobservability subspaces which contain
$$\mathcal{E}_2 = \text{span}\{e_2(x), x \in \mathbb{R}^n\}.$$  \hspace{1cm} (35)

Indeed, once a basis for such a subspace is known, one can directly obtain the corresponding matrices $L$ and $D_i$, $i = 0, \ldots, m$ and a basis for the associated subspace $S(L, D)$. Furthermore a state transformation $T$ that brings the original system into the form Eqs. (15), (16) with $\overline{e}_1^i(\zeta) \equiv 0$ is obtained. It can be shown that the family of $(C,
\( \mathcal{A} \)-unobservability subspaces containing \( \mathcal{E}_2 \), denoted \( U(C, \mathcal{A}; \mathcal{E}_2) \), admits an infimal element,
\[ \mathcal{E}_2^{**}, \tag{36} \]
namely an element which is contained in all the other elements of the family. A necessary and sufficient condition for the existence of a solution to the FPRG is:
\[ \mathcal{E}_1 \subsetneq \mathcal{E}_2^{**}, \tag{37} \]

Geometric system theory provides a computational way to derive a basis for \( \mathcal{E}_2^{**} \). It is based on two sequences of subspaces. The first one:
\[
\begin{align*}
V_0 &= \mathcal{E}_2 \\
V_{i+1} &= V_i + \sum_{j=0}^{m} A_j \left( V_i \cap \ker C \right)
\end{align*}
\tag{38}
\]
is an increasing stationary sequence of which the limit, \( \mathcal{E}_2^{**} \), is the element \( V_k \) such that \( V_k = V_{k'} \) for all \( k \geq k' \). The second one:
\[
\begin{align*}
W_0 &= \mathcal{E}_2^{**} + \ker C \\
W_{j+1} &= \mathcal{E}_2^{**} + \left( \sum_{i=0}^{m} A_i^{-1} W_j \right) \cap \ker C
\end{align*}
\tag{39}
\]
has precisely \( \mathcal{E}_2^{**} \) as a limiting element.

Equations (38) and (39) lead to a numerical algorithm relying only on linear algebraic operations to deduce a basis for \( \mathcal{E}_2^{**} \). From this, the associated matrices \( L^{**} \) and \( D_i^{**}, i = 0, \ldots, m \)
can be determined and the design of a residual generator can be performed.

**Remark 3** There are other approaches to design observer-based residual generators for bilinear systems. Some of them do not resort to the geometric concepts introduced here; however, such tools are at the basis of the generalization of the results to state affine and control affine systems. One of these approaches uses a filter with linear time-invariant error dynamics instead of the time varying dynamics observed in Eq. (31). This makes the implementation of the filter easier. However, there are situations where the FPRG admits a solution, but there is no filter with linear time-invariant error dynamics that solves it, while a filter with linear time-varying error dynamics exists.

**Remark 4** The tendency was to use as few results from nonlinear system theory as possible in the above presentation. Actually, \( S(L, D) \) is linked to the observation space of system Eqs. (7), (8) that will be denoted \( \mathcal{O}(L, D) \). Indeed, the codistribution \( d\mathcal{O}(L, D) = \{d\tau, \tau \in \mathcal{O}\} \) (where \( d \) is the classical differential operator) can be identified with the vector space \( S(L, D) \). Moreover, the
(C, A)-unobservability subspace associated to L and D can be identified with kerdO(L, D).

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**Biographical Sketch**

**Michel Kinnaert** graduated in 1983 from the Université Libre de Bruxelles (ULB), Brussels, Belgium, as a mechanical and electrical engineer. He received his M.S. degree in electrical engineering from Stanford University in 1984, and his Doctor in Applied Sciences from the ULB in 1987. He was employed for 6 years (from 1984 to 1990) by the Belgian National Fund for Scientific Research. In 1987, he was a visiting scientist in the Department of Electrical Engineering and Computer Science of the University of Newcastle, Australia for 6 months. In 1990, he was appointed by the ULB were he is now associate professor in the Department of Control Engineering and System Analysis. He also held two visiting professor positions at the LAGEP in Université Claude Bernard Lyon 1, Lyon, France. Since January 2000, he has been vice-chairman of the IFAC Technical Committee on Fault Detection, Safety and Supervision of Technical Processes. His research interests include fault detection and isolation, and loop monitoring. Both theoretical aspects and applications to the process industry are considered. Another area of his activities is the design of controllers for rolling mills.

**Joseph J. Yamé** graduated in electrical and mechanical engineering from the Université Libre de Bruxelles (ULB), Brussels, Belgium in 1985. He received the post-graduate degree in control engineering and his Doctor of Applied Sciences from the ULB in 1986 and 2001, respectively. In 1987, he was appointed by the National Polytechnic Institute of Ivory Coast (Côte d’Ivoire) as a lecturer in the Electrical and Electronic Engineering Department. Since November 2000, he has been a researcher in the Control Engineering and Systems Analysis Department of the ULB. Over the past years his educational and research activities have focused on different subjects in control engineering with special interests in dual adaptive control of stochastic systems, sampled-data systems theory and their continuous-time behavior, operator theory and infinite-dimensional systems, and the analytical aspects of fuzzy control systems. His recent research interest is mainly concentrated on fault tolerant control systems. He has been a consultant in automation and process control for a variety of industries (breweries, oil refineries, power plants, and others) in the Ivory Coast. He is a member of the American Mathematical Society.