CONSTITUTIVE THEORIES: BASIC PRINCIPLES

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Summary

Constitutive theories deal with mathematical models of material bodies. Aside from physical experiences and experimental observations, theoretical considerations for constitutive equations rest upon some basic principles, among them the most fundamental ones being material frame-indifference and material symmetry. The former states that constitutive equations characterize intrinsic properties of a material body and thereby must be independent of whoever is describing them, while the later requires constitutive equations to comply with symmetry properties of a material body relative to change of reference state. General framework of constitutive formulation consistent with these two requirements is presented and for some classes of material bodies, including elastic solids, viscous fluids and heat conductors, general constitutive equations are analyzed and classical constitutive laws are obtained. Thermodynamic considerations are also important in constitutive theories. It is formulated in the entropy principle which requires constitutive equations to be consistent with the second law of thermodynamics. The exploitation of entropy principle to obtain constitutive restrictions is a major task in rational thermodynamics. As an example, for the class of elastic materials, Gibbs relation of classical thermostatics and Fourier inequality are obtained. Through derivations of the well-known classical results, the presentation in this chapter

emphasizes mathematical implications of basic principles in rational framework of constitutive theories for general material bodies.

1. Introduction

Properties of material bodies are described mathematically by constitutive equations. Classical models, such as Hooke's law of elastic solids, Navier-Stokes law of viscous fluids, and Fourier law of heat conduction, are mostly proposed based on physical experiences and experimental observations. However, even these linear experimental laws did not come without some understanding of theoretical concepts of material behavior. Without it one would neither know what experiments to run nor be able to interpret their results. From theoretical viewpoints, the aim of constitutive theories in continuum mechanics is to construct material models consistent with some universal principles so as to enable us, by formulating and analyzing mathematical problems, to predict the outcomes in material behavior verifiable by experimental observations.

We begin with the concept of observer and its role in describing motion and deformation of a material body in the classical Newtonian space-time. An observer, or a frame of reference, is needed to set the stage for mathematical rendering of describing kinematical behaviors of a body, postulating basic laws of mechanics, and formulating constitutive equations of material models.

For the discussion of material behavior and material properties, the concept of objectivity pertaining to the real nature rather than its value affected by different observers is important. This leads to the first universal principle to be discussed, the principle of material frame-indifference, which simply states that the material properties must be independent of observer. This requirement for constitutive equations will be called the condition of material objectivity.

The second universal principle to be discussed deals with the symmetry properties of a material body. Types of materials, such as fluids, solids, and isotropic materials, can be classified according to their symmetry properties.

The conditions of material symmetry and material objectivity are the two major considerations in deducing the restrictions imposed on constitutive equations for a given class of materials. In particular, classes of elastic solids, viscous heat-conducting fluids, and thermoelastic solids are carefully examined, and the classical laws of Hooke, Navier-Stokes, and Fourier are obtained. However, the derivation of these classical laws, already well-known, is not the main purpose of this presentation; rather it provides the mathematical framework for the basic principles in deducing restrictions imposed on general constitutive equations for material models.

Constitutive theories of materials cannot be complete without some thermodynamic considerations. The entropy principle of continuum thermodynamics requires that constitutive equations be consistent with the entropy inequality for any thermodynamic process, and thus like the conditions of material objectivity and material symmetry, it also imposes restrictions on constitutive equations. Since it is not our purpose to present a comprehensive view of thermodynamics in this chapter, we shall illustrate the basic thermodynamic considerations for elastic materials as an example by a relatively simple procedure for exploiting the entropy principle based on the Clausius-Duhem inequality to obtain constitutive restrictions, such as the Gibbs relation, well-known in classical thermostatics, and the Fourier inequality for heat conduction.

The presentations employ the direct notations commonly used in linear algebra, and only in a few occasions the Cartesian components are used. No mathematics beyond linear algebra and differential calculus are needed to understand the physical ideas and most of derivations in the text. Although this is only a chapter in a sequel of many other topics in continuum mechanics, we try to make it as self-contained as possible.

2. Frame of Reference, Observer

The space-time $\mathcal{W}$ is a four-dimensional space in which physical events occur at some places and certain instants. Let $\mathcal{W}_t$ be the space of placement at the instant t, then the Newtonian space-time of classical mechanics can be expressed as the disjoint union of placement spaces at each instant,

$$\mathcal{W} = \bigcup_{t\in\mathbb{R}} \mathcal{W}_t \ ,$$

and therefore, it can be regarded as a product space $\mathbb{E}\times\mathbb{R}$ through a one-to-one mapping

$$\phi : \mathcal{W} \to \mathbb{E}\times\mathbb{R}, \quad \text{such that} \quad \phi_t : \mathcal{W}_t \to \mathbb{E},$$

where $\mathbb{R}$ is the space of real numbers for time and $\mathbb{E}$ is a three-dimensional Euclidean space.

For a Euclidean space $\mathbb{E}$, there is a vector space $\mathbb{V}$, called the translation space of $\mathbb{E}$, such that the difference $\boldsymbol{v} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ of any two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{E}$ is a vector in $\mathbb{V}$. We also require that the vector space $\mathbb{V}$ be equipped with an inner product, so that length and angle can be defined.

Such a mapping ϕ is called a *frame of reference* and may be regarded as an observer, since it can be depicted as a person taking a snapshot so that the image of ϕ_t is a picture (three-dimensional at least conceptually) of the placements of the events at some instant t, from which the distance between two simultaneous events can be measured. A sequence of events can also be recorded as video clips depicting the change of events in time by an observer.

Now, suppose that two observers are recording the same events with video cameras. In order to compare their video clips regarding the locations and time, they must have a mutual agreement that the clock of their cameras must be synchronized so that simultaneous events can be recognized and since during the recording two observers may move independently while taking pictures with their cameras from different angles, there will be a relative motion and a relative orientation between them. This fact can be expressed mathematically as follows:

Let ϕ and ϕ^* be two observers. We call $* := \phi^* \circ \phi^{-1}$ a *change of frame* (observer),

$$*: \mathbb{E} \times \mathbb{R} \to \mathbb{E} \times \mathbb{R}, \qquad *\left(\boldsymbol{x}, t\right) = \left(\boldsymbol{x}^*, t^*\right),$$

where $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}^*, t^*)$ are the position and time of the same event observed by ϕ and ϕ^* simultaneously. From the observer's agreement, they must be related in the following manner,

$$\boldsymbol{x}^* = Q(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{c}(t), \quad t^* = t + a, \tag{1}$$

for some constant time difference $a \in \mathbb{R}$, some relative translation $c(t) \in \mathbb{E}$ with respect to the reference point $\boldsymbol{x}_0 \in \mathbb{E}$ and some $Q(t) \in \mathcal{O}(\mathbb{V})$, where $\mathcal{O}(\mathbb{V})$ is the group of orthogonal transformations on the translation space $\mathbb{V}$. In other words, a change of frame is an isometry of space and time as well as preserves the sense of time. Such a transformation will be called a *Euclidean transformation*.

In particular, $\phi_t^* \circ \phi_t^{-1} : \mathbb{E} \to \mathbb{E}$ is given by

$$\boldsymbol{x}^* = \phi_t^*(\phi_t^{-1}(\boldsymbol{x})) = Q(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{c}(t), \tag{2}$$

which is a time-dependent rigid transformation consisting of an orthogonal transformation and a translation. We shall often call $Q(t)$ the *orthogonal part* of the change of frame from ϕ to ϕ^*.

3. Motion and Deformation of a Body

In the space-time, the presence of a physical event is represented by its placement at a certain instant so that it can be observed by the observer in a frame of reference. In this sense, the motion of a body can be viewed as a continuous sequence of events such that at any instant t, the placement of the body $\mathcal{B}$ in $\mathcal{W}_t$ is a one-to-one mapping

$$\chi_t : \mathcal{B} \to \mathcal{W}_t.$$

For an observer ϕ with $\phi_t : \mathcal{W}_t \to \mathbb{E}$, the motion can be viewed as a composite mapping $\chi_{\phi_t} := \phi_t \circ \chi_t$,

$$\chi_{\phi_t} : \mathcal{B} \to \mathbb{E}, \quad \boldsymbol{x} = \chi_{\phi_t}(p) = \phi_t(\chi_t(p)), \quad p \in \mathcal{B}.$$

This mapping identifies the body with a region in the Euclidean space, $\mathcal{B}_{\chi_t} := \chi_{\phi_t}(\mathcal{B}) \subset \mathbb{E}$ (see the right part of Figure 1). We call χ_{ϕ_t} a *configuration* of $\mathcal{B}$ at the instant t in the frame ϕ, and a motion of $\mathcal{B}$ is a sequence of configurations of $\mathcal{B}$ in time, $\chi_\phi = \left\{ \chi_{\phi_t}, t \in \mathbb{R} \mid \chi_{\phi_t} : \mathcal{B} \to \mathbb{E} \right\}$. We can also express a motion as

$$\chi_\phi : \mathcal{B} \times \mathbb{R} \to \mathbb{E}, \quad \boldsymbol{x} = \chi_\phi(p,t) = \chi_{\phi_t}(p), \quad p \in \mathcal{B}.$$

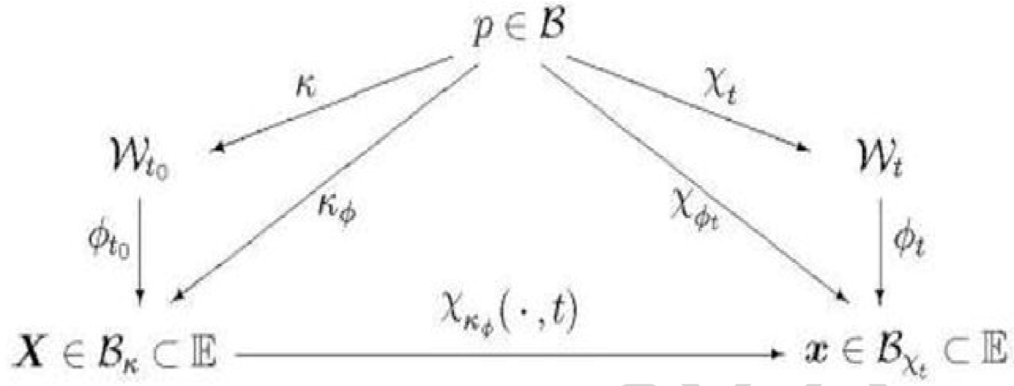

Figure1. Motion χ_{ϕ_t}, reference configuration κ_ϕ and deformation $\chi_{\kappa_\phi}(\cdot, t)$

Reference Configuration

We regard a body $\mathcal{B}$ as a set of material points. Although it is possible to endow the body as a manifold with a differentiable structure and topology for doing mathematics on the body, to avoid such mathematical subtleties, usually a particular configuration is chosen as reference (see the left part of Figure 1),

$$\kappa_\phi : \mathcal{B} \to \mathbb{E}, \quad \boldsymbol{X} = \kappa_\phi(p), \quad \mathcal{B}_\kappa := \kappa_\phi(\mathcal{B}) \subset \mathbb{E},$$

so that the motion at an instant t is a one-to-one mapping

$$\chi_{\kappa_\phi}(\cdot, t) : \mathcal{B}_\kappa \to \mathcal{B}_{\chi_t}, \quad \boldsymbol{x} = \chi_{\kappa_\phi}(\boldsymbol{X}, t) = \chi_{\phi_t}(\kappa_\phi^{-1}(\boldsymbol{X})), \quad \boldsymbol{X} \in \mathcal{B}_\kappa,$$

defined on a domain in the Euclidean space $\mathbb{E}$ for which topology and differentiability are well defined. This mapping is called a *deformation* from κ to χ_t in the frame ϕ and a motion is then a sequence of deformations in time.

Remember that a configuration is a placement of a body relative to an observer, therefore, for the reference configuration κ_ϕ, there is some instant, say t_0, at which the reference placement κ of the body is chosen (see Figure 1). On the other hand, the choice of a reference configuration is arbitrary, and it is not necessary that the body should actually occupy the reference place in its motion under consideration. Nevertheless, in most practical problems, t_0 is usually taken as the initial time of the motion.

We can now define kinematic quantities of the motion with domain in $\mathbb{E} \times \mathbb{R}$, such as,

$$v(\boldsymbol{X},t)=\frac{\partial\chi_{\kappa_\phi}(\boldsymbol{X},t)}{\partial t},\quad \boldsymbol{a}(X,t)=\frac{\partial^2\chi_{\kappa_\phi}(\boldsymbol{X},t)}{\partial t^2},\quad F(\boldsymbol{X},t)=\nabla_{\boldsymbol{X}}\,\chi_{\kappa_\phi}(\boldsymbol{X},t).$$

The velocity $\boldsymbol{v}$ and the acceleration a are vector quantities, $\boldsymbol{v},\boldsymbol{a}\in\mathbb{V}$, while the deformation gradient is a non-singular linear transformation, i.e., $F\in\mathcal{L}(\mathbb{V})$, $\det F\neq 0$, since χ_{κ_ϕ} is a one-to-one mapping in $\mathbb{E}$. The space of linear transformations is denoted by $\mathcal{L}(\mathbb{V})$ and a linear transformation is also called a second order tensor or simply a tensor.

Note that no explicit reference to κ_ϕ is indicated in these quantities for brevity, unless similar notations relative to another frame or another reference configuration are involved.

Here of course, we have assumed that $\chi_{\kappa_\phi}(\boldsymbol{X},t)$ is at least twice differentiable with respect to t and once differentiable with respect to $\boldsymbol{X}$. However, from now on, we shall assume that all functions are smooth enough for the conditions needed in the context, without their smoothness explicitly specified.

Material, Referential and Spatial Descriptions

A material body has some physical properties whose values may change along with the deformation of the body in a motion. A quantity defined on a motion can be described in essentially two different ways: either by the evolution of its value along the trajectory of a material point or by the change of its value at a fixed location in space occupied by the body. The former is called a material description and the later a spatial description. We shall make them more precise below. For simplicity, we shall drop the subscript ϕ relative to a fixed frame in the following discussions.

For a given motion $\boldsymbol{x}=\chi(p,t)$, consider a quantity, with its value in some space $\mathbb{W}$, defined on the motion of $\mathcal{B}$ by a function

$$f:\mathcal{B}\times\mathbb{R}\to\mathbb{W}.$$

Then it can also be defined on a reference configuration κ of $\mathcal{B}$,

$$\hat f:\mathcal{B}_\kappa\times\mathbb{R}\to\mathbb{W},\quad \hat f(\boldsymbol{X},t)=\hat f(\kappa(p),t)=f(p,t).\quad \boldsymbol{X}=\kappa(p),\quad p\in\mathcal{B},$$

and on the position occupied by the body at the present time at time t,

$$\tilde f(\cdot,t):\mathcal{B}_t\to\mathbb{W},\quad \tilde f(\boldsymbol{x},t)=\tilde f\left(\chi_\kappa(\boldsymbol{X},t),t\right)=\hat f(\boldsymbol{X},t),\quad \boldsymbol{x}=\chi_\kappa(\boldsymbol{X},t),\quad \boldsymbol{X}\in\mathcal{B}_\kappa.$$

As a custom in continuum mechanics, one usually denotes these functions f, $\hat f$ and $\tilde f$ by the same symbol since they have the same value at the corresponding point, and write, by an abuse of notations,

$$f = f(p,t) = f(\boldsymbol{X},t) = f(\boldsymbol{x},t),$$

and call them the *material description*, the *referential description* and the *spatial description* of the function f respectively. The referential description is also referred to as the *Lagrangian description* and the spatial description as the *Eulerian description*.

When a reference configuration is chosen and fixed, one can usually identify the material point p with its reference position $\boldsymbol{X}$. In fact, the material description in (p,t) is rarely used and the referential description in $(\boldsymbol{X},t)$ is often regarded as *the* material description instead. However, in later discussions concerning material frame-indifference of constitutive equations, we shall emphasize the difference between the material description and a referential description, because the true nature of material properties should not depend on the choice of a reference configuration.

Possible confusions may arise due to the abuse of notations when differentiations are involved. To avoid such confusions, we shall use different notations for differentiation in these situations:

$$\dot{f} = \frac{\partial f(\boldsymbol{X},t)}{\partial t}, \quad \frac{\partial f}{\partial t} = \frac{\partial f(\boldsymbol{x},t)}{\partial t}, \quad \nabla_X f = \frac{\partial f(\boldsymbol{X},t)}{\partial \boldsymbol{X}}, \quad \nabla_x f = \frac{\partial f(\boldsymbol{x},t)}{\partial \boldsymbol{x}}.$$

The relations between these notations can easily be obtained by the chain rule. Indeed, let f be a scalar field and $\boldsymbol{u}$ be a vector field. We have

$$\dot{f} = \frac{\partial f}{\partial t} + (\nabla_x f)\cdot \boldsymbol{v}, \quad \dot{\boldsymbol{u}} = \frac{\partial \boldsymbol{u}}{\partial t} + (\nabla_x \boldsymbol{u})\boldsymbol{v}, \qquad (3)$$

where v is the velocity of the motion, and the referential and spatial gradients are related by

$$\nabla_X f = F^T(\nabla_x f), \quad \nabla_X \boldsymbol{u} = (\nabla_x \boldsymbol{u})F, \qquad (4)$$

where the superscript T denotes the transpose of a second order tensor. In spatial description, the gradient and the divergence will also be denoted by the usual notations: $(\mathrm{grad}\, f)$, $(\mathrm{grad}\, \boldsymbol{u})$, and $(\mathrm{div}\, \boldsymbol{u})$.

We call $\dot{f}$ the material time derivative of f, which is the time derivative of f following the trajectory of a material point. Therefore, by the use of (3), the velocity v and the acceleration a can be expressed as

$$\boldsymbol{v} = \dot{\boldsymbol{x}}, \quad \boldsymbol{a} = \ddot{\boldsymbol{x}} = \dot{\boldsymbol{v}} = \frac{\partial \boldsymbol{v}}{\partial t} + (\mathrm{grad}\, \boldsymbol{v})\boldsymbol{v}.$$

Moreover, taking the velocity $\boldsymbol{v}$ for $\boldsymbol{u}$ in (4) and since $\nabla_X \boldsymbol{v} = \nabla_X \dot{\boldsymbol{x}} = \dot{F}$, it follows that

$$L := \operatorname{grad} \boldsymbol{v} = \dot{F}F^{-1}, \tag{5}$$

where L is defined as the spatial gradient of the velocity.

4. Objective Tensors

The change of frame (1) on the Euclidean space $\mathbb{E}$ gives rise to a linear mapping on the translation space $\mathbb{V}$, in the following way: Let $\boldsymbol{u}(\phi) = \boldsymbol{x}_2 - \boldsymbol{x}_1 \in \mathbb{V}$ be the difference vector of $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{E}$ in the frame ϕ, and $\boldsymbol{u}(\phi^*) = \boldsymbol{x}_2^* - \boldsymbol{x}_1^* \in \mathbb{V}$ be the corresponding difference vector in the frame ϕ^*, then from (1), it follows immediately that

$$\boldsymbol{u}(\phi^*) = Q(t)\boldsymbol{u}(\phi).$$

Any vector quantity in $\mathbb{V}$, which has this transformation property, is said to be objective with respect to Euclidean transformations, *objective* in the sense that it pertains to a quantity of its real nature rather than its values as affected by different observers. This concept of objectivity can be generalized to any tensor spaces of $\mathbb{V}$.

Definition. *Let s, $\boldsymbol{u}$, and T be scalar-, vector-, (second order) tensor-valued functions respectively. If relative to a change of frame from ϕ to ϕ^*,*

$$\begin{aligned} &s(\phi^*) = s(\phi), \\ &\boldsymbol{u}(\phi^*) = Q(t)\boldsymbol{u}(\phi), \\ &T(\phi^*) = Q(t)\,T(\phi)\,Q(t)^{\mathrm{T}}, \end{aligned}$$

where $Q(t)$ is the orthogonal part of the change of frame from ϕ to ϕ^*, then s, $\boldsymbol{u}$ and T are called objective scalar, vector and tensor quantities respectively.

They are also said to be frame-indifferent with respect to Euclidean transformation or simply Euclidean objective. We call $f(\phi)$ the value of the function observed in the frame ϕ, and for simplicity, often write $f = f(\phi)$ and $f^* = f(\phi^*)$.

The definition of objective scalar is self-evident. For the definition of objective tensors, we consider a scalar $s = \boldsymbol{u}\cdot T\boldsymbol{v}$. Since $(\boldsymbol{u}\cdot T\boldsymbol{v})(\phi) = \boldsymbol{u}(\phi)\cdot T(\phi)\boldsymbol{v}(\phi)$, it follows that

$$\boldsymbol{u}(\phi^*)\cdot T(\phi^*)\boldsymbol{v}(\phi^*) = Q(t)\boldsymbol{u}(\phi)\cdot T(\phi^*)Q(t)\boldsymbol{v}(\phi) = \boldsymbol{u}(\phi)\cdot Q(t)^{\mathrm{T}}T(\phi^*)Q(t)\boldsymbol{v}(\phi),$$

for any objective vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Therefore, if $s = \boldsymbol{u}\cdot T\boldsymbol{v}$ is an objective scalar, that is, $\boldsymbol{u}(\phi^*)\cdot T(\phi^*)\boldsymbol{v}(\phi^*) = \boldsymbol{u}(\phi)\cdot T(\phi)\boldsymbol{v}(\phi)$, then it implies that T is an objective tensor satisfying $T(\phi^*) = Q(t)\,T(\phi)\,Q(t)^{\mathrm{T}}$.

One can easily deduce the transformation properties of functions defined on the position and time under a change of frame. Consider an objective scalar field

$\psi(\boldsymbol{x},t)=\psi^*(\boldsymbol{x}^*,t^*)$. Taking the gradient with respect to x, from (2) we obtain

$$\nabla_{\boldsymbol{x}}\psi(\boldsymbol{x},t)=Q(t)^{\mathrm{T}}\nabla_{\boldsymbol{x}^*}\psi^*(\boldsymbol{x}^*,t^*) \quad \text{or} \quad \left(\operatorname{grad}\psi\right)(\phi^*)=Q(t)\left(\operatorname{grad}\psi\right)(\phi),$$

which proves that $\left(\operatorname{grad}\psi\right)$ is an objective vector field. Similarly, we can show that if $\boldsymbol{u}$ is an objective vector field then $\left(\operatorname{grad}\boldsymbol{u}\right)$ is an objective tensor field and $\left(\operatorname{div}\boldsymbol{u}\right)$ is an objective scalar field. However, one can easily show that the partial derivative $\partial\psi/\partial t$ is not an objective scalar field and neither is $\partial\boldsymbol{u}/\partial t$ an objective vector field.

Transformation Properties of Motion

Let χ_ϕ be a motion of the body in the frame ϕ, and χ_{ϕ^*} be the corresponding motion in ϕ^*,

$$\boldsymbol{x}=\chi_\phi(p,t), \quad \boldsymbol{x}^*=\chi_{\phi^*}(p,t^*), \quad p\in\mathcal{B}.$$

Then from (2), we have

$$\chi_{\phi^*}(p,t^*)=Q(t)(\chi_\phi(p,t)-\boldsymbol{x}_0)+\boldsymbol{c}(t), \quad p\in\mathcal{B},$$

from which, one can easily show that the velocity and the acceleration are not objective quantities,

$$\dot{\boldsymbol{x}}^*=Q\dot{\boldsymbol{x}}+\dot{Q}(\boldsymbol{x}-\boldsymbol{x}_0)+\dot{\boldsymbol{c}},$$

$$\ddot{\boldsymbol{x}}^*=Q\ddot{\boldsymbol{x}}+2\dot{Q}\dot{\boldsymbol{x}}+\ddot{Q}(\boldsymbol{x}-\boldsymbol{x}_0)+\ddot{\boldsymbol{c}}. \tag{6}$$

A change of frame (1) with constant $Q(t)$ and $\boldsymbol{c}(t)=\boldsymbol{c}_0+\boldsymbol{c}_1 t$, for constant $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$, is called a *Galilean transformation*. Therefore, from (6) we conclude that the acceleration is not Euclidean objective but it is frame-indifferent with respect to Galilean transformation. Moreover, it also shows that the velocity is neither a Euclidean nor a Galilean objective vector quantity.

Transformation Properties of Deformation Gradient

Let $\kappa:\mathcal{B}\to\mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0 (see Figure 2), then

$$\kappa_\phi=\phi_{t_0}\circ\kappa \quad \text{and} \quad \kappa_{\phi^*}=\phi^*_{t_0}\circ\kappa \tag{7}$$

are the corresponding reference configurations of $\mathcal{B}$ in the frames ϕ and ϕ^* at the same instant, and

$$\boldsymbol{X} = \kappa_\phi(p), \quad \boldsymbol{X}^* = \kappa_{\phi^*}(p), \quad p \in \mathcal{B}.$$

Let us denote by $\gamma = \kappa_{\phi^*} \circ \kappa_\phi^{-1}$ the change of reference configuration from κ_ϕ to κ_{ϕ^*} in the change of frame, then it follows from (7) that $\gamma = \phi^*_{t_0} \circ \phi^{-1}_{t_0}$ and by (2), we have

$$\boldsymbol{X}^* = \gamma(\boldsymbol{X}) = K(\boldsymbol{X} - \boldsymbol{x}_0) + \boldsymbol{c}(t_0), \tag{8}$$

where $K = Q(t_0)$ is a constant orthogonal tensor.

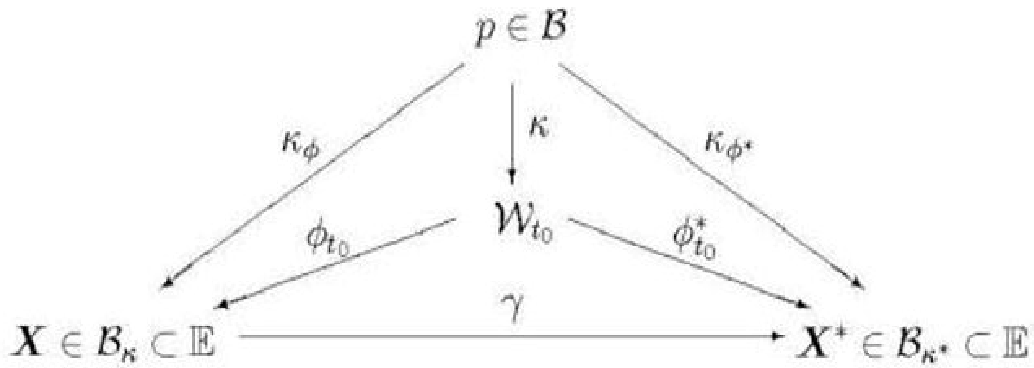

Figure2. Reference configurations κ_ϕ and κ_{ϕ^*} in the change of frame from ϕ to ϕ^*

On the other hand, the motion in referential description relative to the change of frame is given by $\boldsymbol{x} = \chi_\kappa(\boldsymbol{X}, t)$ and $\boldsymbol{x}^* = \chi_{\kappa^*}(\boldsymbol{X}^*, t^*)$. Hence from (2), we have

$$\chi_{\kappa^*}(\boldsymbol{X}^*, t^*) = Q(t)(\chi_\kappa(\boldsymbol{X}, t) - \boldsymbol{x}_0) + \boldsymbol{c}(t).$$

Therefore we obtain for the deformation gradient in the frame ϕ^*, i.e., $F^* = \nabla_{\boldsymbol{X}^*} \chi_{\kappa^*}$, by the chain rule and the use of (8),

$$F^*(\boldsymbol{X}^*, t^*) = Q(t) F(\boldsymbol{X}, t) Q(t_0)^{\mathrm{T}}, \quad \text{or simply} \quad F^* = QFK^{\mathrm{T}}, \tag{9}$$

where $K = Q(t_0)$ is a constant orthogonal tensor due to the change of frame for the reference configuration.

Remark. The transformation property (9) stands in contrast to $F^* = QF$, the widely used formula which is obtained “provided that the reference configuration is unaffected by the change of frame” as usually implicitly assumed, so that K reduces to the identity transformation.

The deformation gradient F is not a Euclidean objective tensor. However, the property (9) also shows that it is frame-indifferent with respect to Galilean transformations, since in this case, $K = Q$ is a constant orthogonal transformation.

From (9), we can easily obtain the transformation properties of other kinematic quantities associated with the deformation gradient. In particular, let us consider the velocity gradient defined in (5). We have

$$L^* = \dot{F}^*(F^*)^{-1} = (Q\dot{F} + \dot{Q}F)K^{\mathrm{T}}(QFK^{\mathrm{T}})^{-1} = (Q\dot{F} + \dot{Q}F)F^{-1}Q^{\mathrm{T}},$$

which gives

$$L^* = QLQ^{\mathrm{T}} + \dot{Q}Q^{\mathrm{T}}. \tag{10}$$

Moreover, with the decomposition $L = D + W$ into symmetric and skew-symmetric parts, it becomes

$$D^* + W^* = Q(D+W)Q^{\mathrm{T}} + \dot{Q}Q^{\mathrm{T}}.$$

By separating symmetric and skew-symmetric parts, we obtain

$$D^* = QDQ^{\mathrm{T}}, \quad W^* = QWQ^{\mathrm{T}} + \dot{Q}Q^{\mathrm{T}},$$

since $\dot{Q}Q^{\mathrm{T}}$ is skew-symmetric. Therefore, while the velocity gradient L and the *spin tensor* W are not objective, the *rate of strain tensor* D is an objective tensor.

5. Galilean Invariance of Balance Laws

Motivated by classical mechanics, the balance laws of mass, linear momentum, and energy for deformable bodies,

$$\begin{aligned} &\dot{\rho} + \rho \operatorname{div} \dot{\boldsymbol{x}} = 0, \\ &\rho \ddot{\boldsymbol{x}} - \operatorname{div} T = \rho \boldsymbol{b}, \\ &\rho \dot{\varepsilon} + \operatorname{div} \boldsymbol{q} - T \cdot \operatorname{grad} \dot{\boldsymbol{x}} = \rho r, \end{aligned} \tag{11}$$

in an inertial frame must be invariant under Galilean transformation. Since by definition, two inertial frames are related by a Galilean transformation, it means that the equations (11) should hold in the same form in any inertial frame. In particular, the balance of linear momentum takes the forms in the inertial frames ϕ and ϕ^*,

$$\rho \ddot{\boldsymbol{x}} - \operatorname{div} T = \rho \boldsymbol{b}, \quad \rho^* \ddot{\boldsymbol{x}}^* - (\operatorname{div} T)^* = \rho^* \boldsymbol{b}^*.$$

Since the acceleration $\ddot{\boldsymbol{x}}$ is Galilean invariant, in order this to hold, it is usually assumed that the mass density ρ, the Cauchy stress tensor T and the body force $\boldsymbol{b}$ are objective scalar, tensor, and vector quantities respectively. Similarly, for the energy equation, it is also assumed that the internal energy ε and the energy supply r are objective scalars, and the heat flux $\boldsymbol{q}$ is an objective vector. These assumptions concern the non-kinematic quantities, including external supplies $(\boldsymbol{b}, r)$, and the constitutive quantities $(T, \boldsymbol{q}, \varepsilon)$.

In fact, for Galilean invariance of the balance laws, only frame-indifference with respect to Galilean transformation for all those non-kinematic quantities would be sufficient. However, traditionally based on physical experiences, it is *postulated* that they are not only Galilean objective but also Euclidean objective. Therefore, with the known transformation properties of the kinematic variables, the balance laws in any arbitrary frame can be deduced.

To emphasize the importance of the objectivity postulate for constitutive theories, it will be referred to as Euclidean objectivity for constitutive quantities:

Euclidean objectivity. *The constitutive quantities: the Cauchy stress T, the heat flux $\boldsymbol{q}$ and the internal energy density ε, are objective (frame-indifferent with respect to Euclidean transformation),*

$$T(\phi^*) = Q(t)\, T(\phi)\, Q(t)^{\mathrm{T}}, \quad \boldsymbol{q}(\phi^*) = Q(t)\, \boldsymbol{q}(\phi), \quad \varepsilon(\phi^*) = \varepsilon(\phi), \tag{12}$$

where $Q(t)$ is the orthogonal part of the change of frame from ϕ to ϕ^*.

Note that this postulate concerns only objective properties of balance laws, so that it is a universal property for any material bodies.

6. Constitutive Equations in Material Description

Physically a state of the thermomechanical behavior of a body is characterized by a description of the fields of density $\rho(p,t)$, motion $\chi(p,t)$ and temperature $\theta(p,t)$. The material properties of a body generally depend on the past history of its thermomechanical behavior.

Let us introduce the notion of the past history of a function. Let $h(\cdot)$ be a function of time. The history of h up to time t is defined by

$$h^t(s) = h(t-s),$$

where $s \in [0, \infty)$ denotes the time-coordinate pointed into the past from the present time t. Clearly $s = 0$ corresponds to the present time, therefore $h^t(0) = h(t)$.

Mathematical descriptions of material properties are called constitutive equations. We postulate that the history of thermomechanical behavior up to the present time determines the properties of the material body.

Principle of determinism. *Let ϕ be a frame of reference, and $\mathcal{C}$ be a constitutive quantity, then the constitutive equation for $\mathcal{C}$ is given by a functional of the form,*

$$\mathcal{C}(p,t) = \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t; p), \quad p \in \mathcal{B}, t \in \mathbb{R}, \tag{13}$$

where the first three arguments are history functions:

$$\rho^t : \mathcal{B} \times [0,\infty) \to \mathbb{R}, \quad \chi^t : \mathcal{B} \times [0,\infty) \to \mathbb{E}, \quad \theta^t : \mathcal{B} \times [0,\infty) \to \mathbb{R}.$$

We call $\mathcal{F}_\phi$ the constitutive function of $\mathcal{C}$ in the frame ϕ. Such a function allows the description of arbitrary non-local effect of an inhomogeneous body with a perfect memory of the past thermomechanical history. With the notation $\mathcal{F}_\phi$, we emphasize that the value of a constitutive function may depend on the observer (frame of reference ϕ).

Constitutive equations can be regarded as mathematical models of material bodies. For a general and rational formulation of constitutive theories, asides from physical experiences, one should rely on some basic principles that a mathematical model should obey lest its consequences be contradictory to physical nature. The most fundamental principles are

- Principle of material frame-indifference,
- Material symmetry,
- Entropy principle.

We shall see in the following sections that these principles impose severe restrictions on material models and hence lead to a great simplification for general constitutive equations. The reduction of constitutive equations to more specific and mathematically simpler ones for a given class of materials is the main objective of constitutive theories in continuum mechanics.

Condition of Euclidean Objectivity

Let $\mathcal{C} = \{T, \boldsymbol{q}, \varepsilon\}$ be constitutive quantities and ϕ be a frame of reference. Then from (13), the constitutive equation can be written as

$$\mathcal{C}(p,t) = \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t; p), \quad p \in \mathcal{B}, \ t \in \mathbb{R}. \tag{14}$$

Similarly, relative to the frame ϕ^*, the corresponding constitutive equation can be written as

$$\mathcal{C}^*(p,t^*) = \mathcal{F}_{\phi^*}((\rho^t)^*, (\chi^t)^*, (\theta^t)^*, t^*; p), \quad p \in \mathcal{B}, \ t^* \in \mathbb{R}. \tag{15}$$

The two constitutive functions $\mathcal{F}_\phi$ and $\mathcal{F}_{\phi^*}$ are not independent, they must satisfy the Euclidean objectivity relation (12). In particular, for the stress, it implies that

$$\mathcal{F}_{\phi^*}((\rho^t)^*, (\chi^t)^*, (\theta^t)^*, t^*; p) = Q(t)\,\mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t; p)\,Q(t)^{\mathrm{T}}, \quad p \in \mathcal{B}, \tag{16}$$

for any histories $s \in [0,\infty)$ and $p' \in \mathcal{B}$, such that

$$(\rho^t)^*(p',s) = \rho^*(p',t^*-s) = \rho(p',t-s),$$

$$(\theta^t)^*(p',s) = \theta^*(p',t^*-s) = \theta(p',t-s), \qquad (17)$$

$$(\chi^t)^*(p',s) = Q(t-s)(\chi^t(p',s)-\boldsymbol{x}_o) + \boldsymbol{c}(t-s),$$

where $Q(t) \in \mathcal{O}(\mathbb{V})$, $\boldsymbol{x}_o, \boldsymbol{c}(t) \in \mathbb{E}$ are associated with the change of frame from ϕ to ϕ^*.

The first two relations of (17) state that the density and the temperature are objective scalar fields and the last relation follows from the change of frame given by (2).

The relation (16) will be referred to as the *condition of Euclidean objectivity*. It is a relation between the constitutive functions relative to two different frames, and indeed, it determines the constitutive function $\mathcal{F}_{\phi^*}$ once the constitutive function $\mathcal{F}_\phi$ is given.

Example. Consider a material model given by the constitutive equation,

$$T = \mathcal{T}_\phi(\rho, L) = \alpha(\rho)\, I + \beta(\rho)\, L,$$

where I is the identity tensor and L is the velocity gradient. By Euclidean objectivity, $\rho^* = \rho$, $T^* = QTQ^{\mathrm{T}}$, and from the transformation formula (10), $L^* = QLQ^{\mathrm{T}} + \dot{Q}Q^{\mathrm{T}}$, it follows that

$$T^* = Q\,\mathcal{T}_\phi(\rho, L)\,Q^{\mathrm{T}} = \alpha(\rho^*)\, I + \beta(\rho^*)(L^* - \dot{Q}Q^{\mathrm{T}}) := \mathcal{T}_{\phi^*}(\rho^*, L^*),$$

where the constitutive function $\mathcal{T}_{\phi^*}$, defined in the above relation, depends explicitly on $Q(t)$ of the change of frame from ϕ to ϕ^*. From this example, it shows that in general the constitutive functions may depend on the relation between the observers.

7. Principle of Material Frame-Indifference

It is obvious that *not* any proposed constitutive equations can be used as material models. First of all, they may be frame-dependent in general. However, since the constitutive functions must characterize the intrinsic properties of the material body itself, it should be independent of observer. Consequently, there must be some restrictions imposed on the constitutive functions so that they would be indifferent to the change of frame. This is the essential idea of the principle of material frame-indifference.

Principle of material frame-indifference (in material description)**.***The constitutive function of an objective (frame-indifferent with respect to Euclidean transformations)* *constitutive quantity $\mathcal{C}$ must be independent of frame, i.e., for any frames of reference ϕ and ϕ^*, the functionals $\mathcal{F}_\phi$ and $\mathcal{F}_{\phi^*}$, defined by (14) and (15), must have the same form,*

$$\mathcal{F}_\phi(\bullet; p) = \mathcal{F}_{\phi^*}(\bullet; p), \quad p \in \mathcal{B}. \qquad (18)$$

where $\bullet$ represents the same arguments in both functionals.

This will be referred to as *form-invariance,* $\mathcal{F}_\phi(\bullet) = \mathcal{F}_{\phi^*}(\bullet)$, while in (18) the material point p is superfluously indicated to emphasize that it is valid only when the material description is used. The implication of form invariance (18) in referential descriptions will be considered in the next section. Moreover, for a non-objective constitutive quantity, such as the total energy (the kinetic part is not objective), since the quantity itself is frame-dependent, it is obvious that its constitutive function can never be independent of frame.

In our discussions, we shall often consider only the constitutive function of stress, an objective tensor quantity, for simplicity. Similar results can easily be obtained for any other vector or scalar objective constitutive quantities. Thus, from the condition of Euclidean objectivity (16) and the principle of material frame-indifference (18), we obtain the following condition:

Condition of material objectivity. *The constitutive function of an objective tensor constitutive quantity, in material description, satisfies the condition,*

$$\mathcal{F}_\phi((\rho^t)^*, (\chi^t)^*, (\theta^t)^*, t^*; p) = Q(t)\,\mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t; p)\,Q(t)^{\mathrm{T}}, \quad p \in \mathcal{B}, \qquad (19)$$

for any histories related by (17 where the change of frame $*$ is arbitrary.

Since the condition (19) involves only the constitutive function in the frame ϕ, it becomes a restriction imposed on the constitutive function $\mathcal{F}_\phi$.

We emphasize that in the condition of Euclidean objectivity (16), $Q(t)$ is the orthogonal part of the change of frame from ϕ to ϕ^*. However, the condition of material objectivity is valid for an arbitrary change of frame from ϕ. Therefore, the condition (19) is valid for any $Q(t) \in \mathcal{O}(\mathbb{V})$.

Sometimes, the condition of material objectivity is referred to as the "*principle* of material objectivity", to impart its relevance in characterizing material property and Euclidean objectivity, as a more explicit form of the principle of material-frame indifference. Indeed, the original principle of material frame-indifference in the fundamental treatise by Truesdell and Noll (1965) was formulated in the form (19).

An immediate restriction imposed by the condition of material objectivity can be obtained by considering a change of frame given by ($Q(t)=1,\ \boldsymbol{c}(t)=\boldsymbol{x}_0$)

$$\boldsymbol{x}^*=\boldsymbol{x}, \quad t^*=t+a,$$

for arbitrary constant $a\in\mathbb{R}$. By (17), the condition (19) implies that

$$\mathcal{F}_\phi(\rho^t,\chi^t,\theta^t,t+a;p)=\mathcal{F}_\phi(\rho^t,\chi^t,\theta^t,t;p).$$

Since this is true for any value of $a\in\mathbb{R}$, we conclude that F_ϕ can not depend on the argument t and the constitutive equations in general can be expressed as

$$\mathcal{C}(p,t)=\mathcal{F}_\phi(\rho^t,\chi^t,\theta^t;p), \quad p\in\mathcal{B},\ t\in\mathbb{R}. \tag{20}$$

Example. In the Example of the previous section, it shows that the constitutive function $T=\mathcal{T}_\phi(\rho,L)$ is frame-dependent. We shall determine the restriction imposed on the constitutive function $\mathcal{T}_\phi$ so that the condition of material objectivity (19) is satisfied. The condition (19) for $\mathcal{T}_\phi$ becomes

$$Q\mathcal{T}_\phi(\rho,L)Q^{\mathrm{T}}=\mathcal{T}_\phi(\rho,QLQ^{\mathrm{T}}+\dot{Q}Q^{\mathrm{T}})=\mathcal{T}_\phi(\rho,QDQ^{\mathrm{T}}+QWQ^{\mathrm{T}}+\dot{Q}Q^{\mathrm{T}}), \tag{21}$$

which is valid for any orthogonal transformation $Q(t)$, where D and W are the symmetric and skew-symmetric parts of the velocity gradient L. Let us consider a transformation $Q^t(s)=Q(t-s)$ such that it satisfies the following differential equation and the initial condition:

$$\dot{Q}^t(s)+WQ^t(s)=0, \quad Q^t(0)=I.$$

It is known that the solution exists and can be expressed as $Q^t(s)=\exp(-sW)$ which is an orthogonal transformation because W is skew-symmetric. Clearly it implies that

$$Q(t)=Q^t(0)=I, \quad \dot{Q}(t)=\dot{Q}^t(0)=-W.$$

Therefore, for this choice of orthogonal transformation $Q(t)$ the relation (21) implies that

$$\mathcal{T}_\phi(\rho,L)=\mathcal{T}_\phi(\rho,D).$$

In other words, the condition (19) imposes the restriction that the dependence of the function $\mathcal{T}_\phi$ on L must reduces to the dependence only on the symmetric part of L.

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Biographical Sketch

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