FULL-ORDER STATE OBSERVERS

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Contents

Summary

An observer is a dynamic system $\hat{\mathcal{S}}$ the purpose of which is to estimate the state of another dynamic system $\mathcal{S}$ using only the measured input and output of the latter. If the order of $\hat{\mathcal{S}}$ is equal to the order of $\mathcal{S}$ the observer is said to be "full-order"; if the order of $\hat{\mathcal{S}}$ is less than the order of $\mathcal{S}$ the observer is "reduced order." (See *Reduced Order State Observers*)

A full-order observer accomplishes its purpose by calculating the "residual," which is the difference between the measured output and the corresponding quantity generated by a model of $\mathcal{S}$ synthesized in the observer. The residual, multiplied by a "gain" is used as an input to a model of $\mathcal{S}$. If the gain is chosen appropriately, the observer $\hat{\mathcal{S}}$ will be an asymptotically stable dynamic system and the estimation error will converge to zero.

If the observer gain is optimized for the noise input to $\mathcal{S}$ and to the sensor(s), the observer is called a "Kalman filter." If the gain is not so optimized, the observer may be

termed a “Luenberger observer”.

The original theory of observers, as developed by Kalman and by Luenberger, was concerned only with linear dynamic systems. Many applications, however, required observers for nonlinear systems, and extensions to the linear theory have been developed during the years following the appearance of the original theory.

1. Introduction

There are many situations in the modern technology in which it is necessary to estimate the *state* of a dynamic system using only the measured input and output data of the system. If the system is *observable* (See *Description and Analysis of Dynamic Systems in State Space*) it is possible to achieve this goal, i.e., to determine the state $\mathbf{x}(t)$ of a dynamic system by suitably processing the input $\mathbf{u}(\tau)$ and output $\mathbf{y}(\tau)$ of the system for $\tau \in [t, T]$ where $T > t$ is sufficiently large. The procedure for determining $\mathbf{x}(t)$ is not unique.

One method of achieving the desired result is the use of an *observer*. An observer for a dynamic system $\mathcal{S}(\mathbf{x}, \mathbf{y}, \mathbf{u})$ with state $\mathbf{x}$, output $\mathbf{y}$, and input $\mathbf{u}$ is another dynamic system $\hat{\mathcal{S}}(\hat{\mathbf{x}}, \mathbf{y}, \mathbf{u})$ having the property that the state $\hat{\mathbf{x}}$ of the observer $\hat{\mathcal{S}}$ converges to the state $\mathbf{x}$ of the process $\mathcal{S}$, independent of the input $\mathbf{u}$ or the state $\mathbf{x}$. The initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ of the process $\mathcal{S}$ is assumed to be unknown, hence the initial state $\hat{\mathbf{x}}_0$ of the observer is an estimate of $\mathbf{x}_0$.

Among the various applications for observers, perhaps the most important is for the implementation of closed-loop control algorithms designed by state space methods. The control algorithm is designed in two parts: a “full-state feedback” part based on the assumption that all the state variables can be measured; and an observer to estimate the state of the process based upon the observed output. The concept of separating the feedback control design into these two parts is known as the *separation principle* which has rigorous validity in linear systems and in a limited class of nonlinear systems. Even when its validity cannot be rigorously established, the separation principle is often a practical solution to many design problems.

The concept of an observer for a dynamic process was introduced in 1966 by D. Luenberger. The generic “Luenberger observer,” however, appeared several years after the Kalman filter, which is in fact an important special case of a Luenberger observer—an observer optimized for the noise present in the observations and in the input to the process.

2. Linear Observers

2.1. Continuous-Time Systems

Consider a linear, continuous- time dynamic system

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

$$\mathbf{y} = \mathbf{Cx}. \tag{2}$$

The more generic output

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

can be treated by defining a modified output

$$\bar{\mathbf{y}} = \mathbf{y} - \mathbf{Du}$$

and working with $\bar{\mathbf{y}}$ instead of $\mathbf{y}$. (The direct coupling $\mathbf{Du}$ from the input to the output is absent in many physical plants.).

A full-order observer for the linear process defined by (1) and (2) has the generic form

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \mathbf{Ky} + \mathbf{Hu}, \tag{3}$$

where the dimension of state $\hat{\mathbf{x}}$ of the observer is equal to the dimension of process state $\mathbf{x}$.

The matrices $\hat{\mathbf{A}}, \mathbf{K}$, and $\mathbf{H}$ appearing in (3) must be chosen to conform with the required property of an observer: that the observer state must converge to the process state independent of the state $\mathbf{x}$ and the input $\mathbf{u}$. To determine these matrices, let

$$\mathbf{e} := \mathbf{x} - \hat{\mathbf{x}} \tag{4}$$

be the estimation error. From (1), (2), and (3)

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{Ax} + \mathbf{Bu} - \hat{\mathbf{A}}(\mathbf{x} - \mathbf{e}) - \mathbf{KCx} - \mathbf{Hu} \\ &= \hat{\mathbf{A}}\mathbf{e} + (-\hat{\mathbf{A}} + \mathbf{A} - \mathbf{KC})\mathbf{x} + (\mathbf{B} - \mathbf{H})\mathbf{u}.\end{aligned} \tag{5}$$

From (5) it is seen that for the error to converge to zero independent of $\mathbf{x}$ and $\mathbf{u}$, the following conditions must be satisfied:

$$\hat{\mathbf{A}} = \mathbf{A} - \mathbf{KC} \tag{6}$$

$$\mathbf{H} = \mathbf{B}. \tag{7}$$

When these conditions are satisfied, the estimation error is governed by

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e}, \tag{8}$$

which converges to zero if $\hat{\mathbf{A}}$ is a “stability matrix”, i.e., that (8) is an asymptotically stable dynamic system. When $\hat{\mathbf{A}}$ is constant, this means that its eigenvalues must lie in the (open) left half plane.

Note that the initial state of (8) is

$$\mathbf{e}_0 = \mathbf{x}_0 - \hat{\mathbf{x}}_0$$

hence, if the initial state of the process under observation is known precisely (i.e., $\mathbf{x}_0 = \hat{\mathbf{x}}_0$) then the estimation error is zero thereafter. Due to the possibility of the occurrence of disturbances (not necessarily the “white noise” assumed in the Kalman filter), however, the true state $\mathbf{x}$ may depart from the solution to (1) during the course of operation of the observer. Hence knowledge of the initial state $\mathbf{x}(t_0)$ does not eliminate the need for an observer in practical situations.

Since the matrices $\mathbf{A}, \mathbf{B},$ and $\mathbf{C}$ are defined by the plant, the only freedom in the design of the observer is in the selection of the gain matrix $\mathbf{K}$.

To emphasize the role of the observer gain matrix, and accounting for requirements of (6) and (7), the observer can be written as

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}). \tag{9}$$

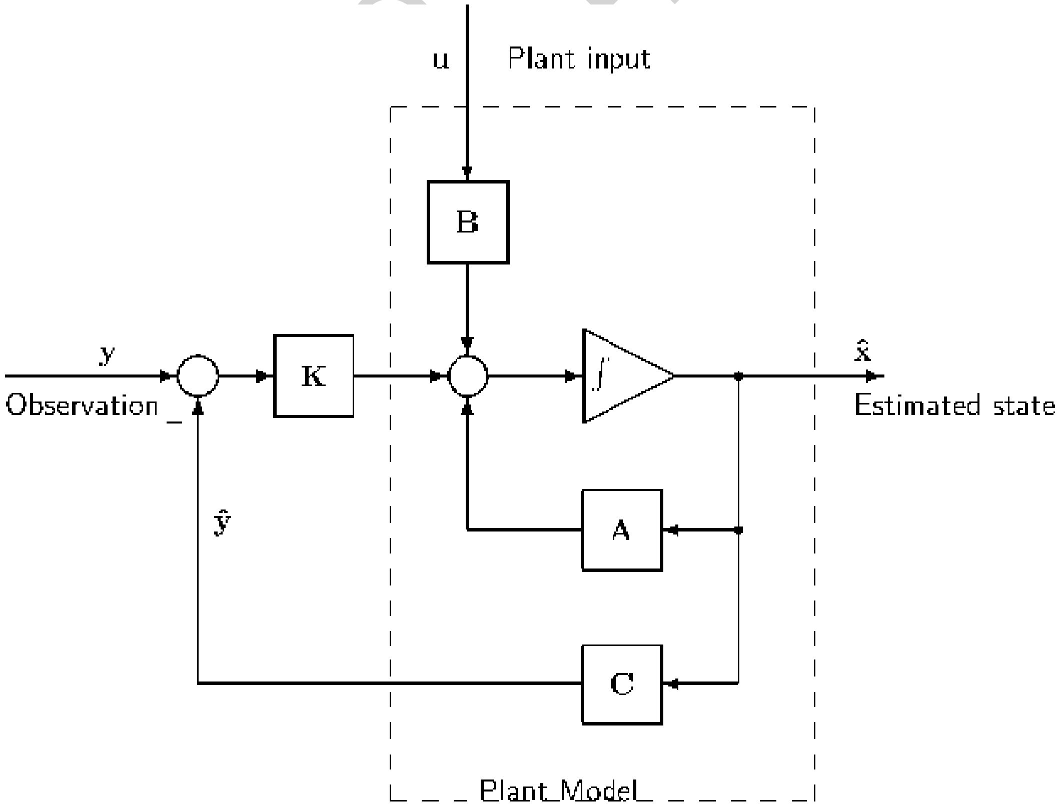

Figure 1: Full-order observer for linear process.

A block-diagram representation of (9), as given in Figure 1, aids in the interpretation of the observer. Note that the observer comprises a model of the process with an added input:

$\mathbf{K}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) = \mathbf{K}\mathbf{r}$.

The quantity

$$\mathbf{r} := \mathbf{y} - \mathbf{C}\hat{\mathbf{x}} = \mathbf{y} - \hat{\mathbf{y}} \tag{10}$$

often called the *residual*, is the difference between the actual observation $\mathbf{y}$ and the "synthesized" observation

$$\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}$$

produced by the observer. The observer can be viewed as a feedback system designed to drive the residual to zero: as the residual is driven to zero, the input to (9) due to the residual vanishes and the state of (9) looks like the state of the original process.

The fundamental problem in the design of an observer is the determination of the observer gain matrix $\mathbf{K}$ such that the closed-loop observer matrix

$$\hat{\mathbf{A}} = \mathbf{A} - \mathbf{K}\mathbf{C} \tag{11}$$

is a stability matrix, as defined above.

There is considerable flexibility in the selection of the observer gain matrix. Two methods are standard: optimization, and pole-placement.

2.1.1 Optimization

Since the observer given by (9) has the structure of a Kalman filter, (see *Kalman Filters*.) its gain matrix can be chosen as a Kalman filter gain matrix, i.e.,

$$\mathbf{K} = \mathbf{P}\mathbf{C}'\mathbf{R}^{-1}, \tag{12}$$

where $\mathbf{P}$ is the covariance matrix of the estimation error and satisfies the matrix Riccati equation

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}' - \mathbf{P}\mathbf{C}'\mathbf{R}^{-1}\mathbf{C}\mathbf{P} + \mathbf{Q}, \tag{13}$$

where $\mathbf{R}$ is a positive-definite matrix and $\mathbf{Q}$ is a positive, semi-definite matrix. The matrices $\mathbf{R}$ and $\mathbf{Q}$ are, respectively, the spectral density matrices of the white noise processes driving the observation (the "observation noise") and the system dynamics (the "process noise").

The initial condition on (13)

$$\mathbf{P}_0 = \mathbf{P}(t_0)$$

is the initial state covariance matrix is chosen to reflect the uncertainty of the state at the starting time t_0.

In many applications the steady-state covariance matrix is used in (12). This matrix is given by setting $\dot{\mathbf{P}}$ in (13) to zero. The resulting equation is known as the *algebraic Riccati equation—ARE*. Algorithms to solve the ARE are included in popular control system software packages such as Matlab.

In order for the gain matrix given by (12) and (13) to be genuinely optimum, the process noise and the observation noise must be white with the matrices $\mathbf{Q}$ and $\mathbf{R}$ being their spectral densities. It is rarely possible to determine these spectral density matrices in practical application. Hence, the matrices $\mathbf{Q}$ and $\mathbf{R}$ can be treated as design parameters which can be varied to achieve overall system design objectives.

If the observer is to be used as a state estimator in a closed-loop control system, an appropriate form for the matrix $\mathbf{Q}$ is

$$\mathbf{Q} = q^2\mathbf{B}\mathbf{B}'. \tag{14}$$

As has been shown by Doyle and Stein, as $q \to \infty$, this observer tends to “recover” the stability margins assured by a full-state feedback control law obtained by quadratic optimization.

2.1.2. Pole-Placement

An alternative to solving the algebraic Riccati equation to obtain the observer gain matrix is to select $\mathbf{K}$ to place the poles of the observer, i.e., the eigenvalues of $\hat{\mathbf{A}}$ in (11). (See *Pole Placement Control*.)

When there is a single observation, $\mathbf{K}$ is a column vector with exactly as many elements as eigenvalues of $\hat{\mathbf{A}}$. Hence specification of the eigenvalues of $\hat{\mathbf{A}}$ uniquely determines the gain matrix $\mathbf{K}$. A number of algorithms can be used to determine the gain matrix, some of which are incorporated into the popular control system design software packages. Some of the algorithms have been found to be numerically ill-conditioned, so caution should be exercised in using the results.

The author of this chapter has found the Bass-Gura formula effective in most applications. This formula gives the gain matrix as

$$\mathbf{K} = (\mathbf{O}\mathbf{W})'^{-1}(\hat{\mathbf{a}} - \mathbf{a}), \tag{15}$$

where

$$\mathbf{a} = [a_1 \quad a_2 \quad \dots \quad a_n]' \tag{16}$$

is the vector formed from the coefficients of the characteristic polynomial of the process matrix $\mathbf{A}$:

$$| s\mathbf{I} - \mathbf{A} | = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n \tag{17}$$

and $\hat{a}$ is the vector formed from the coefficients of the desired characteristic polynomial

$$| s\mathbf{I} - \hat{\mathbf{A}} | = s^n + \hat{a}_1 s^{n-1} + \dots + \hat{a}_{n-1} s + \hat{a}_n \,. \tag{18}$$

The other matrices in (15) are given by

$$\mathbf{O} = [\mathbf{C}' \quad \mathbf{A}'\mathbf{C}' \quad \dots \quad \mathbf{A}'^{n-1}\mathbf{C}'], \tag{19}$$

which is the *observability matrix* of the process, and

$$\mathbf{W} = \begin{bmatrix} 1 & a_1 & \cdots & a_n \\ 0 & 1 & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{20}$$

The determinant of $\mathbf{W}$ is 1, so it is not singular. If the observability matrix $\mathbf{O}$ is not singular, the inverse matrix required in (15) exists. Hence the gain matrix $\mathbf{K}$ can be found which places the observer poles at arbitrary locations if (and only if) the process for which an observer is sought is observable.

Ackermann's algorithm (cited by Kailath and incorporated in the Matlab suite) is an alternative to the Bass-Gura algorithm.

Numerical problems occur with both the Bass-Gura algorithm and the Ackermann algorithm, when the observability matrix is nearly singular. Other numerical problems can arise in determination of the characteristic polynomial $| s\mathbf{I} - \mathbf{A} |$ for high order systems and in the determination of $s\mathbf{I} - \hat{\mathbf{A}}$ when the individual poles, and not the characteristic polynomial, are specified. In such instances, it may be necessary to use an algorithm designed to handle difficult numerical calculations, such as the algorithm of Kautsky and Nichols, which is included in the Matlab suite.

When two or more quantities are observed, there are more elements in the gain matrix than eigenvalues of $\hat{\mathbf{A}}$, so specification of the eigenvalues of $\hat{\mathbf{A}}$ does not uniquely specify the gain matrix $\mathbf{K}$. In addition to placing the eigenvalues, more of the "eigenstructure" of $\hat{\mathbf{A}}$ can be specified. This method of selecting the gain matrix is fraught with difficulty, however, and the use of the algebraic Riccati equation is usually preferable. The Kautsky-Nichols algorithm can also deal with more than a single observation input. It uses the additional degrees of freedom afforded by the multiple input to achieve enhanced robustness in the observer.

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Biographical Sketch

Dr. Bernard Friedland is a Distinguished Professor in the Department of Electrical and Computer Engineering at the New Jersey Institute of Technology which he joined in January 1990. He was a Lady Davis Visiting Professor at the Technion--Israel Institute of Technology and has held appointments as an Adjunct Professor of Electrical Engineering at the Polytechnic University, New York University, and Columbia University. He was born and educated in New York City and received his B.S., M.S., and Ph.D. degrees from Columbia University.

Dr. Friedland is author of two textbooks on automatic control and co-author of two other textbooks: one on circuit theory and the other on linear system theory. He is the author or co-author of over 100 technical papers on control theory and its applications. His theoretical contributions include: a technique of quasi-optimum control, treatment of bias in recursive filtering, design of reduced-order linear regulators, modeling of pulse-width modulated control systems, maximum likelihood failure detection, friction modeling and compensation, and parameter estimation.

For 27 years prior to joining NJIT, Dr. Friedland was Manager of Systems Research in the Kearfott Guidance and Navigation Corporation. While at Kearfott, he was awarded 12 patents in the field of navigation, instrumentation, and control systems.

Dr. Friedland is the recipient of the 1982 Oldenberger Medal of the ASME. He is a Fellow of the IEEE, and has received the the IEEE Third Millennium Medal and the Control Systems Society's Distinguished Member Award. He is also a Fellow of the ASME.