REDUCED-ORDER STATE OBSERVERS

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Contents

Summary

A reduced-order observer for a dynamic process S is a dynamic process of order $q = n - m$, where n is the order of S and m is the number of (independent) observations. In addition to being more parsimonious of state variables, the reduced order observer may exhibit performance superior to that of a full-order observer, particularly in a closed-loop control system.

1. Introduction

An observer is a dynamic system $\hat{S}$ the purpose of which is to estimate the state of another dynamic system S using only the measured input and the measured output of the latter. If the order of $\hat{S}$ is equal to the order of S the observer is said to be "full-order" (see *Full Order State Observers*.); if the order of $\hat{S}$ is less than the order of S the observer is "reduced order".

Because the number of state variables in a reduced-order observer is less than the order n of S by the number m of (independent) observations, the reduced-order observer is parsimonious, often a desirable engineering quality. But, in addition, a reduced-order observer may have better properties than a full-order observer, especially with regard to robustness of a control system which uses an observer to implement the control algorithm in an "observer-based" design.

2. Linear, Reduced-Order Observers

The theory of reduced-order observers is simplified by partitioning the state vector into two substates:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \cdots \\ \mathbf{x}_2 \end{bmatrix} \tag{1}$$

such that

$$\mathbf{x}_1 = \mathbf{y} = \mathbf{C}\mathbf{x} = \hat{\mathbf{x}}_1 \tag{2}$$

where

$$\mathbf{C} = [\mathbf{I} \quad \mathbf{0}]$$

is the observation vector (of dimension m) and $\mathbf{x}_2$ (of dimension $n-m$) comprises the components of the state vector that cannot be measured directly.

The assumption that $\mathbf{y} = \mathbf{x}_2$ makes the resulting equations simpler, but it is not necessary. Equivalent results can be obtained for any observation matrix $\mathbf{C}$ of rank m.

In terms of $\mathbf{x}_1$ and $\mathbf{x}_2$ the plant dynamics are written

$$\dot{\mathbf{x}}_1 = \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2 + \mathbf{B}_1\mathbf{u} \tag{3}$$

$$\dot{\mathbf{x}}_2 = \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{x}_2 + \mathbf{B}_2\mathbf{u} \tag{4}$$

Since $\mathbf{x}_1$ is directly measured, no observer is required for that substate, i.e.

$$\hat{\mathbf{x}}_1 = \mathbf{x}_1 = \mathbf{y} \tag{5}$$

For the remaining substate, we define the reduced-order observer by

$$\hat{\mathbf{x}}_2 = \mathbf{K}\mathbf{y} + \mathbf{z} \tag{6}$$

where $\mathbf{z}$ is the state of a system of order $n-m$:

$$\dot{\mathbf{z}} = \hat{\mathbf{A}}\mathbf{z} + \mathbf{L}\mathbf{y} + \mathbf{H}\mathbf{u} \tag{7}$$

A block-diagram representation of the reduced-order observer is given in Figure 1(a)

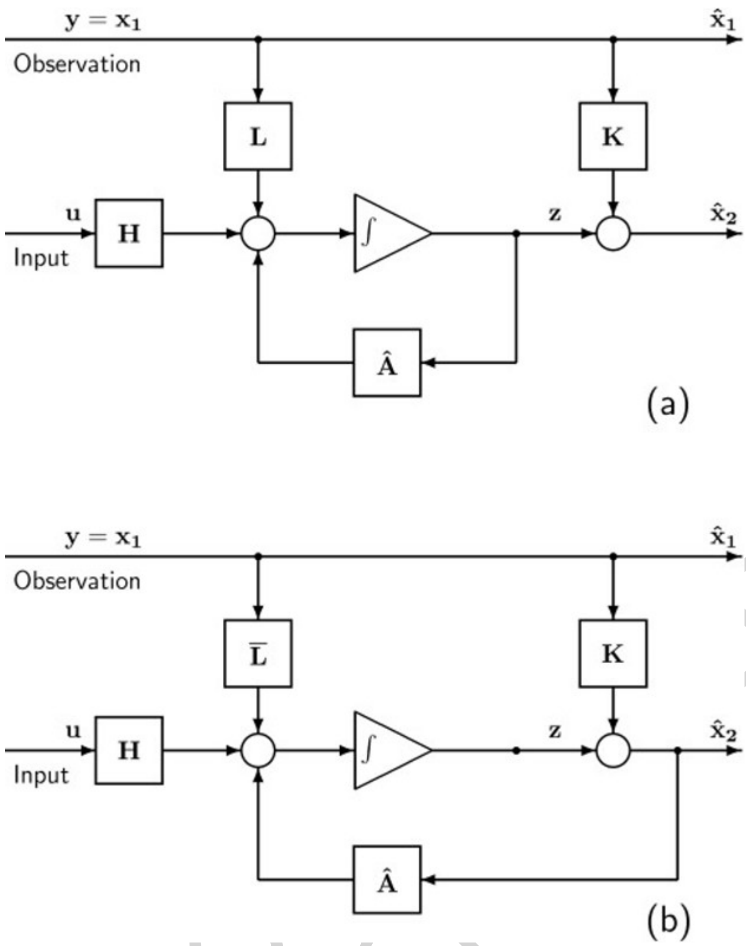

Figure 1: Reduced-order observer (Two forms).

The matrices $\hat{\mathbf{A}}, \mathbf{L}, \mathbf{H}$, and $\mathbf{K}$ are chosen, as in the case of the full-order observer, to ensure that the error in the estimation of the state converges to zero, independent of $\mathbf{x}, \mathbf{y},$ and $\mathbf{u}$.

Since there is no error in estimation of $\mathbf{x}_1$, i.e.,

$$\mathbf{e}_1 = \mathbf{x}_1 - \hat{\mathbf{x}}_1 = \mathbf{0} \tag{8}$$

by virtue of (5), it is necessary only to ensure the convergence of

$$\mathbf{e}_2 = \mathbf{x}_2 - \hat{\mathbf{x}}_2 \tag{9}$$

to zero.

From (4) – (7)

$$\dot{\mathbf{e}}_2 = \left(\mathbf{A}_{21} - \mathbf{K}\mathbf{A}_{11} + \hat{\mathbf{A}}\mathbf{K} - \mathbf{L}\right)\mathbf{x}_1 + \left(\mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12} - \hat{\mathbf{A}}\right)\mathbf{x}_2 + \hat{\mathbf{A}}\mathbf{e}_2 + \left(\mathbf{B}_2 - \mathbf{K}\mathbf{B}_1 - \mathbf{H}\right)\mathbf{u}$$

(10)

As in the case of the full-order observer, to make the coefficients of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{u}$ vanish it is necessary that the matrices in (5) and (7) satisfy

$$\hat{\mathbf{A}} = \mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12} \tag{11}$$

$$\mathbf{L} = \mathbf{A}_{21} - \mathbf{K}\mathbf{A}_{11} + \hat{\mathbf{A}}\mathbf{K} \tag{12}$$

$$\mathbf{H} = \mathbf{B}_2 - \mathbf{K}\mathbf{B}_1 \tag{13}$$

Two of these conditions (11) and (13) are analogous to conditions for a full-order observer; (12) is a new requirement for the additional matrix $\mathbf{L}$ that is required by the reduced-order observer.

When these conditions are satisfied, the error in estimation of $\mathbf{x}_2$ is given by

$$\mathbf{e}_2 = \hat{\mathbf{A}}\mathbf{e}_2$$

Hence the gain matrix $\mathbf{K}$ must be chosen such that the eigenvalues of $\hat{\mathbf{A}} = \mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12}$ lie in the (open) left-half plane; $\mathbf{A}_{22}$ and $\mathbf{A}_{12}$ in the reduced-order observer take the roles of $\mathbf{A}$ and $\mathbf{C}$ in the full-order observer; once the gain matrix $\mathbf{K}$ is chosen, there is no further freedom in the choice of $\mathbf{L}$ and $\mathbf{H}$.

The specific form of the new matrix $\mathbf{L}$ in (12) suggests another option for implementation of the dynamics of the reduced-order observer, namely:

$$\dot{\mathbf{z}} = \hat{\mathbf{A}}\hat{\mathbf{x}}_2 + \bar{\mathbf{L}}\mathbf{y} + \mathbf{H}\mathbf{u} \tag{14}$$

where

$$\bar{\mathbf{L}} = \mathbf{A}_{21} - \mathbf{K}\mathbf{A}_{11} \tag{15}$$

A block-diagram representation of this option is given in Figure 1 (b)

The selection of the gain matrix $\mathbf{K}$ of the reduced-order observer may be accomplished by any of the methods that can be used to select the gains of the full-order observer as discussed in the previous article. In particular, pole-placement, using any convenient algorithm is feasible.

The gain matrix can also be obtained as the solution of a reduced-order Kalman filtering problem, taking into account the cross-correlation between the observation noise and the process noise. Suppose the dynamic process is governed by

$$\dot{\mathbf{x}}_1 = \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2 + \mathbf{B}_1\mathbf{u} + \mathbf{F}_1 v \tag{16}$$

$$\dot{\mathbf{x}}_2 = \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{x}_2 + \mathbf{B}_2\mathbf{u} + \mathbf{F}_2 v \tag{17}$$

with the observation being noise-free:

$$\mathbf{y} = \mathbf{x}_1 \tag{18}$$

In this case the gain matrix is given by

$$\mathbf{K} = \left(\mathbf{P}\mathbf{A}_{12}' + \mathbf{F}_2\mathbf{Q}\mathbf{F}_1'\right)\mathbf{R}^{-1} \tag{19}$$

where

$$\mathbf{R} = \mathbf{F}_1\mathbf{Q}\mathbf{F}_1'$$

and $\mathbf{P}$ is the $(n-m)\times(n-m)$ covariance matrix of the estimation error $\mathbf{e}_2$, as given by

$$\dot{\mathbf{P}} = \tilde{\mathbf{A}}\mathbf{P} + \mathbf{P}\tilde{\mathbf{A}}' - \mathbf{P}\mathbf{A}_{12}\mathbf{R}^{-1}\mathbf{A}_{12}'\mathbf{P} + \tilde{\mathbf{Q}} \tag{20}$$

where

$$\tilde{\mathbf{A}} = \mathbf{A}_{22} - \mathbf{F}_2\mathbf{Q}\mathbf{F}_1'\mathbf{R}^{-1}\mathbf{A}_{12} \tag{21}$$

$$\tilde{\mathbf{Q}} = \mathbf{F}_2\mathbf{Q}\mathbf{F}_2' - \mathbf{F}_2\mathbf{Q}\mathbf{F}_1'\mathbf{R}^{-1}\mathbf{F}_1\mathbf{Q}\mathbf{F}_2' \tag{22}$$

The initial condition on (20) is

$$\mathbf{P}\left(t_0\right) = \mathbf{P}_0$$

the covariance matrix of the initial uncertainty of the substate $\mathbf{x}_2$.

Note that (20) becomes homogeneous when

$$\tilde{\mathbf{Q}} = \mathbf{0} \tag{23}$$

In this case it is possible that

$$\lim_{t\to\infty} \mathbf{P}(t) := \mathbf{P}(\infty) = \mathbf{0} \tag{24}$$

which means that the steady-state error in estimating $\mathbf{x}_2$ converges to zero! We can't expect to achieve anything better than this. Unfortunately, $\mathbf{P}(\infty) = \mathbf{0}$ is not the only possible steady-state solution to (23). To test whether it is, it is necessary to check whether the eigenvalues of the resulting observer dynamics matrix

$$\hat{\mathbf{A}} = \mathbf{A}_{22} - \mathbf{F}_2\mathbf{F}_1^{-1}\mathbf{A}_{12} \tag{25}$$

lie in the open left half-plane. If not, (24) is not the correct steady-state solution to (20).

The eigenvalues of the "zero steady-state variance" observer dynamics matrix (25) have an interesting interpretation: as shown in by Friedland in 1989 (among others) these eigenvalues are the transmission zeros of the plant with respect to the noise input to the process. Hence the variance of the estimation error converges to zero if the plant is "minimum phase" with respect to the noise input.

For purposes of robustness the noise distribution matrix $\mathbf{F}$ should include a term proportional to the control distribution matrix $\mathbf{B}$, i.e.,

$$\mathbf{F} = \overline{\mathbf{F}} + q^2\mathbf{B}\mathbf{B}'$$

In this case, the zero-variance observer gain would satisfy

$$\mathbf{H} = \mathbf{B}_2 - \mathbf{K}\mathbf{B}_1 = \mathbf{0} \tag{26}$$

as $q \to \infty$.

If (26) is satisfied the observer poles are located at the transmission zeros of the plant. Thus, in order to use the gain givenby (26), it is necessary for the plant to be minimum-phase with respect to the input. In 1982 Rynaski has defined observers meeting this requirement as *robust observers* which, as discussed below, have remarkable robustness characteristics.

When a reduced-order observer is used, it is readily established that the closed-loop dynamics are given by

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{G} & \mathbf{B}\mathbf{G}_2 \\ \mathbf{0} & \mathbf{A}_{22} - \mathbf{K}\mathbf{A}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e}_2 \end{bmatrix} \tag{27}$$

and hence that the eigenvalues of the closed-loop system are given by

$$|s\mathrm{I} - \mathbf{A} + \mathbf{B}\mathbf{G}||s\mathrm{I} - \mathbf{A}_{22} + \mathbf{K}\mathbf{A}_{12}| = 0 \tag{28}$$

Thus the separation principle also holds when a reduced-order observer is used.

Robustness can be assessed by carrying out an analysis for a reduced-order observer similar to the analysis for a full-order observer. It is found that the characteristic polynomial for the closed-loop control system, when a reduced-order observer is used and the actual control distribution matrix $\mathbf{B} = \overline{\mathbf{B}} + \delta\mathbf{B}$ differs from the nominal (design) value $\overline{\mathbf{B}}$, is given by

$$\left|s\mathbf{I} - \mathbf{A}\right| = \begin{vmatrix} s\mathbf{I} - \mathbf{F} + \Delta\mathbf{G}_2 & \Delta\mathbf{G} \\ -\mathbf{B}\mathbf{G}_2 & s\mathbf{I} - \mathbf{A}_c \end{vmatrix} \qquad (29)$$

where

$$\Delta = \mathbf{K}\delta\mathbf{B}_1 - \mathbf{B}_2 \qquad (30)$$

It is seen that the characteristic polynomial of the closed-loop system reduces to that of (28) when

$$\Delta = 0 \qquad (31)$$

It is noted that (31) can hold in a single-input system in which the loop gain is the only variable parameter. In this case

$$\delta\mathbf{B}_1 = \rho\mathbf{B}_1, \quad \delta\mathbf{B}_2 = \rho\mathbf{B}_2 \qquad (32)$$

and thus

$$\Delta = \rho\left(\mathbf{K}\mathbf{B}_1 - \mathbf{B}_2\right) = -\rho\mathbf{H}$$

Hence, if the observer is designed to satisfy (26) $\left(\mathbf{H} = \mathbf{B}_2 - \mathbf{K}\mathbf{B}_1 = 0\right)$ the separation principle holds for arbitrary changes in the loop gain, thus justifying Rynaski's terminology.

If the system is such that (26) cannot be satisfied, then, as shown by Madiwale and Williams, an condition analogous to the full order Doyle-Stein condition can be derived from (29) in the case of a scalar control input. The condition is

$$\left[\mathbf{I} - \mathbf{K}\left(\mathbf{I} + \mathbf{A}_{12}\mathbf{\Phi}_{22}\mathbf{K}\right)^{-1}\mathbf{A}_{12}\mathbf{\Phi}_{22}\right]\left(\mathbf{B}_2 - \mathbf{K}\mathbf{B}_1\right) = 0 \qquad (33)$$

where

$$\mathbf{\Phi}_{22} = \left(s\mathbf{I} - \mathbf{A}_{22}\right)^{-1}$$

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Biographical Sketch

Bernard Friedland is a Distinguished Professor in the Department of Electrical and Computer Engineering at the New Jersey Institute of Technology which he joined in January 1990. He was a Lady Davis Visiting Professor at the Technion--Israel Institute of Technology and has held appointments as an Adjunct Professor of Electrical Engineering at the Polytechnic University, New York University, and Columbia University. He was born and educated in New York City and received his B.S., M.S., and Ph.D. degrees from Columbia University.

Dr. Friedland is author of two textbooks on automatic control and co-author of two other textbooks: one on circuit theory and the other on linear system theory. He is the author or co-author of over 100 technical papers on control theory and its applications. His theoretical contributions include: a technique of quasi-optimum control, treatment of bias in recursive filtering, design of reduced-order linear regulators, modeling of pulse-width modulated control systems, maximum likelihood failure detection, friction modeling and compensation, and parameter estimation.

For 27 years prior to joining NJIT, Dr. Friedland was Manager of Systems Research in the Kearfott Guidance and Navigation Corporation. While at Kearfott, he was awarded 12 patents in the field of navigation, instrumentation, and control systems.

Dr. Friedland is the recipient of the 1982 Oldenberger Medal of the ASME. He is a Fellow of the IEEE, and has received the IEEE Third Millennium Medal and the Control Systems Society's Distinguished Member Award. He is also a Fellow of the ASME.