STATE ESTIMATION IN DISTRIBUTED PARAMETER SYSTEMS

Vande Wouwer A.
Faculté Polytechnique de Mons, Belgium

Zeitz M.
University of Stuttgart, Germany

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Summary

Many processes from science and engineering are distributed parameter systems (DPSs), that is, they are represented by partial differential equations (PDEs) and boundary conditions (BCs) that describe the temporal and spatial variations of the state variables. The state estimation problem is, given the available on-line measurement information and a dynamic model of the process, to compute the best estimate of the state variables at the current time. This problem is particularly acute in DPSs, since complete spatial profiles of the state variables are usually nonmeasurable and have to be inferred from a limited set of pointwise measurements. System observability and state estimate convergence are strongly influenced by the sensor configuration (i.e., number and location of the measurement sensors), which must be carefully investigated.

Basically, two major approaches to the state estimation problem in DPSs can be taken:

- Early lumping, in which the PDEs of the process model are first approximated in space (e.g., using finite difference or weighted residual methods), and conventional state estimation techniques—such as Luenberger observers or Kalman filters—are applied to the resulting lumped system.

- Late lumping, in which the distributed nature of the process is kept as long as possible in the observer/filter design procedure, and approximation techniques are applied at the final stage only, in order to compute a solution to the observer/filter PDEs.

Early lumping is a straightforward procedure which, however, suffers from the resulting problem dimensionality and the lack of physical interpretation. Late lumping is, therefore, the preferred approach in this article, which reviews several known methods for linear and nonlinear state estimator design, including the extensions of the Luenberger observer and Kalman filter concepts to DPSs. Attention is then focused on a practical, heuristic method to design nonlinear distributed parameter observers, which is based on a physical interpretation of the state estimation error PDE and the injection of correction terms in the process model PDEs and BCs in order to ensure the state estimate convergence. The main limitation of this method is the lack of rigorous analysis of the observer convergence, which is not tractable in the general nonlinear case. Therefore, the observer convergence must be investigated carefully in simulation.
Finally, the application of linear and nonlinear distributed parameter (DP) observers is illustrated with a number of case studies reported in the literature, which demonstrate the efficiency and wide applicability of this state estimation technique.

1. Introduction

Advanced process monitoring and control usually require information on all the state variables. However, the knowledge of the system state variables is limited by the number of sensors, the time delay in processing the measurements, and the noise corrupting the data. The state estimation problem is—given this limited measurement information and a dynamic model of the process—to compute the best estimate of the state variables at the current time. This problem occurs in lumped parameter systems (LPSs), but is even more acute in distributed parameter systems (DPSs), since complete spatial profiles of the state variables are usually nonmeasurable and have to be inferred from a limited set of pointwise measurements. This is especially true in the case of multidimensional systems, where measurements are usually available on the boundary surface only, and the interior state profiles have to be estimated from these surface measurements. For instance, such estimation problems arise in thermal and chemical engineering processes if component concentrations are unavailable because of the lack of appropriate sensors (or long processing times), and the temperature distribution is measured at a few points only. Mechanical structures exhibit the same kind of situations if deflection or acceleration are known at a few spatial locations only (see *Distributed Parameter Systems*).

The solution to the state estimation problem for DPSs is based on a mathematical model of the process, which consists of partial differential equations (PDEs), boundary conditions (BCs), as well as sensor or output equations (see *Partial Differential Equations*). Thereby, model parameters and disturbances are assumed to be known or to be modeled by additional equations. Most PDE models are derived from first principles, and are given in a natural state space representation to which observability analysis and design of state estimation schemes directly apply.

Owing to the infinite order of DPSs and the different classes of PDE models, care must be exercised in designing a Kalman filter or a Luenberger observer. In the course of the design procedure, an early or late lumping approach can be used. In the early lumping approach, the model partial differential equations are approximated first, using for instance modal decomposition, finite difference, or finite element techniques, while the state estimator design proceeds with the approximated model equations. In the late lumping approach, the distributed nature of the system is kept as long as possible, and a state estimation scheme is formulated using PDEs and BCs. In either case, the implementation of the state estimation algorithm requires, sooner or later, the spatial approximation of PDEs and BCs, leading to a system of ordinary differential equations (ODEs) and algebraic equations (AEs) to be solved numerically.

As known from LPS theory, the state estimation problem is much more difficult to address in the nonlinear case. In first-principles models, the nonlinearities often reflect important phenomena such as reaction rates, adsorption kinetics, and heat transfer by radiation. DPSs are then described by nonlinear PDEs and BCs. Neither the early lumping nor the late lumping approach enables an exact solution to the state estimation problem. The most widely used nonlinear state estimation technique for LPSs (i.e., the extended Kalman filter (EKF)), can be applied in its original form to the discretized model PDEs. However, this leads to a high-order LPS that lacks a physical interpretation of the many design parameters. Alternatively, the EKF can be generalized for DPSs, resulting in a set of PDEs that includes an (at least) two-dimensional (in space) Riccati equation.

In the DPS literature, state estimation is formulated in a rather comprehensive and general manner. Somewhat disappointingly, however, there is still a lack of practical methods that could be applied satisfactorily to yet unexplored state estimation problems in science and engineering. On the other hand, tremendous progresses in mathematical modeling and numerical simulation of DPSs can be observed, which provide optimal conditions for the application of on-line state estimation techniques. In this connection, the authors have developed a heuristic approach to the design of DP observers, which is based on the injection of correction functions in the model PDEs and BCs in order to ensure convergence of the state estimates. This design procedure is of general applicability for linear and nonlinear DPSs, and has been tested successfully in simulation and in real-case studies, including various processes such as furnaces, tubular and circulation-loop fixed-bed reactors, distillation columns, and adsorption processes.

The outline of this contribution follows the sequence of solution steps to the state estimation problem. In the next section, the state space representation of DPSs is introduced, in both the deterministic and the stochastic cases. In addition, a brief account of DPS observability and the selection of optimal sensor locations is given. The discussion is restricted to practical results, which can be useful in designing a DP state estimator. Section 3 is devoted to optimal estimation and Kalman filtering. Thereby, a distinction is made between the early lumping approach and the late lumping approach to the design of extended Kalman filters for nonlinear DPSs. In Section 4, Luenberger's concept of state observer is extended to DPSs. The design of the observer correction terms follows a late lumping approach and is described for linear and nonlinear systems.

Finally, the application of linear and nonlinear DP observers is illustrated with a number of case studies reported in the literature, which demonstrate the efficiency and wide applicability of this state estimation technique.

2. State Estimation Problem

In order to introduce the state estimation problem for DPSs, it is natural to consider the analogous problem for finite dimensional linear time-invariant LPSs first:

$$\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t) + B\mathbf{u}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{1}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) \tag{2}$$

where the matrices $A \in \Re^{n\times n}$, $B \in \Re^{n\times p}$, and $C \in \Re^{m\times n}$ are constant.

This classical problem can be formulated as follows: is it possible, based on model (1)–(2), to reconstruct the state **x**(t) from the input **u**(t) and the output measurements **y**(t)? The answer to this question is affirmative if the system is observable. This property can be checked numerically using algebraic conditions on A and C (e.g., a rank test on the observability matrix $[C^T \;\; A^T C^T \ldots (A^T)^{n-1} C^T]$).

Linear time-invariant DPSs can also be described by an abstract state space representation according to (1) and (2) in Hilbert space. In this case, the state **x**(z,t), the input **u**(z,t), and the output **y**(z,t) do not only depend on time t but also on a spatial coordinate z. Then, A, B, and C are differential or integral operators with respect to z. This abstract representation allows an extension of the observability conditions derived for LPSs to DPSs.

In practice, however, DPSs are usually nonlinear and are described by first-principles models. In this latter case, a rigorous analysis becomes difficult, and physical as well as mathematical properties of the DPS model have to be used for observability assessment as well as state estimator design.

2.1. State Space Model

Based on first principles (e.g., mass, energy, and momentum balances), modeling of DPSs results in a natural state space representation, which, in many cases, consists of a system of parabolic or hyperbolic PDEs in the form:

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{x}_t = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{x}_z, \mathbf{x}_{zz}, \ldots) \qquad z \in (0, L) \tag{3}$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{x}_z, \ldots) \qquad z \in \{0, L\} \tag{4}$$

$$\mathbf{x}(z,0)=\mathbf{x_0}(z) \qquad z\in[0,L] \tag{5}$$

For the sake of simplicity, a one-dimensional space domain with fixed boundaries is considered. In PDEs (3) and BCs (4), $\mathbf{x}_z$ and $\mathbf{x}_{zz}$ represent first- and second-order partial derivatives of the state **x**(z,t) with respect to the spatial coordinate z. In (3), **f** is a vector of functions of the state and several of its spatial derivatives as well as of the input **u**(z,t), which is distributed over the length L of the DPS. At the boundaries z = 0 and z = L, the balance equations reduce to algebraic BCs (4). There, the function vector **g** depends on the boundary inputs **u**(0,t) and **u**(L,t) at z = 0 and z = L, respectively. Note that, in some applications, these model PDEs and BCs are supplemented by additional ODEs and/or AEs (e.g., ODEs describing a LPS interacting with the DPS). Moreover, it is assumed that the DPS model (3)–(5) is well posed and has a unique solution.

For example, if x(z,t) represents the temperature profile T(z,t) of a rod isolated at both ends, PDE (3) and BCs (4) could take the form:

$$x_t=(\lambda/\rho c)\, x_{zz}+(4\alpha/d\rho c)(u-x) \quad z\in(0,1) \tag{6}$$

$$0=x_z \qquad z\in\{0,1\} \tag{7}$$

where d is the diameter of the rod (small with respect to normalized length L = 1, so that the temperature distribution is assumed to be uniform in the cross-section), λ is the heat conductivity, ρc is the heat capacity, α is the heat transfer coefficient, and u(z,t) is a cooling or heating profile over z.

The incomplete knowledge of the initial profile $\mathbf{x_0}$(z) in (5) is the main reason for the existence of a state estimation problem. Otherwise, the complete state profile **x**(z,t) at the current time t could be determined by numerical solution of the model Eqs. (3)–(5). However, state reconstruction based on a pure simulation also requires exact knowledge of all the functions and parameters in PDEs (3) and BCs (4). This assumption excludes modeling uncertainties and the presence of disturbances in the DPS.

Consideration of these problems leads to the design of state estimators that combine in an appropriate way the process model (3)–(5) with the available measurement information. Under practical conditions, it can be assumed that the state **x** or functions h_i(**x**) of the state, respectively, are measured in spatial points z_i, i = 1, …, m. The corresponding output or sensor equations define the output or measurement vector:

$$\mathbf{y}(t)=[h_1(\mathbf{x}(z_1,t)),....,h_m(\mathbf{x}(z_m,t))]^T \qquad z_i\in[0,L] \tag{8}$$

In many practical applications, the sensor model (8) reduces to pointwise measurements of some of the state variables x_i, i = 1, …, n at points z_j, j = 1, …, m (e.g., $h_j(\mathbf{x}(z_j,t)) = x_i(z_j,t)$). In examples (6) and (7), the temperature could be measured in m points distributed along the rod, that is:

$$y_i(t)=T(z_i,t) \qquad z_i\in[0,L] \quad \text{i = 1, …, m} \tag{9}$$

Eqs. (3)–(5) and (8) constitute a deterministic representation of the DPS and the associated measurements. Under these assumptions (i.e., neither process disturbances nor measurement noise), the solution of the state estimation problem requires the design of an observer with distributed parameters. On the other hand, if the DPS as well as the measurements are corrupted by noise, which can be modeled by additive zero-mean, normally distributed, white noise vectors δ(z,t), and **ε**(t) entering in the right-hand side of Eqs. (3), (4), and (8), respectively:

$$\mathbf{x}_t(z,t) = \mathbf{f}(\mathbf{x}(z,t), \mathbf{u}(z,t), \mathbf{x}_z(z,t), \mathbf{x}_{zz}(z,t), ...) + \boldsymbol{\delta}(z,t) \qquad z \in (0,L) \tag{10}$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{x}_z, ...) + \boldsymbol{\delta}(z,t) \qquad z \in \{0, L\} \tag{11}$$

$$\mathbf{y}(t) = [h_1(\mathbf{x}(z_1,t)), ..., h_m(\mathbf{x}(z_m,t))]^T + \boldsymbol{\varepsilon}(t) \qquad z_i \in [0,L] \tag{12}$$

then a stochastic state estimation problem has to be solved. In this case, a Kalman filter with distributed parameters is used to reconstruct the state **x**(z,t) of the DPS.

In practice, the measurements are often obtained at discrete times t_k (k = 1,2, …) only, so that Eq. (12) becomes:

$$\mathbf{y}(t_k) = [h_1(\mathbf{x}(z_1,t_k)), ..., h_m(\mathbf{x}(z_m,t_k))]^T + \boldsymbol{\varepsilon}(t_k) \qquad z_i \in [0,L] \tag{13}$$

2.2. Observability and Optimal Sensor Location

The design of a state estimator assumes system observability, a property that is closely related to the selection of an appropriate sensor configuration (i.e., number and location of the measurement sensors in the spatial domain under consideration). More precisely, a DPS is said to be observable if every initial state **x**(**z**,0) can be determined based on the knowledge of the system input **u**(**z**,t) and measurement vector **y**(t) over some finite time interval [0,T]. The derivation of observability conditions for certain classes of PDE systems has been the subject of very active research. For instance, for first-order hyperbolic systems, the observability conditions require that each characteristic line intersects a sensor and that a lumped observability condition is satisfied along these characteristic lines. For second-order linear partial differential equations, observability conditions can be stated in an approximate way using modal analysis (i.e., decomposition in terms of the system eigenfunctions) (see *Controllability and Observability of Distributed Parameter Systems*).

Unfortunately, for general nonlinear DPSs, formal observability results are much more difficult to derive, and it is necessary to resort to some kind of approximation and/or heuristic approach to assess the system observability.

One obvious possibility is to use linearization and spatial discretization techniques (e.g., finite differences, finite elements), in order to transform the original DPS into a (usually high-order) linear LPS to which classical observability analysis directly applies. However, the properties of nonlinear systems can only be assessed locally through linearization. Furthermore, sensor location criteria based on the condition number of the observability matrix of the linearized and discretized model depend on the size of the discretized state vector as well as on the discretization technique. Hence, these criteria are in some instances difficult to interpret, or even subject to numerical errors.

Besides this seemingly natural approach, other more practically oriented procedures have been proposed, such as the selection of a sensor configuration that optimizes the convergence of the state estimator. In particular, the sensor configuration can be selected so as to minimize an optimality criterion based on the error covariance matrix of a Kalman filter. The main drawback of this approach is that it requires the design of a filter prior to the selection of the sensor number and locations, and an iterative procedure involving a relatively lengthy manipulation of error covariance matrices in order to determine the “optimal” sensor configuration.

A basic approach to the selection of a sensor configuration, which has proved very useful in many applications, is based on the knowledge of the dynamic behavior of the DPS, which can be investigated in simulation. If realistic scenarios are considered for the DPS simulation, the visualization of spatial profiles can be used for the selection of the number and location of the sensors in the DPS. However, the simple consideration of the numerical values taken by the state variables in the measurement points can be misleading. In fact, it is interesting to maximize the information content conveyed by the measurement sensors and to minimize potential redundancies. In this connection, a test of independence between the sensor responses is proposed to determine the spatial regions where the sensors should be located.

More specifically, let $\mathbf{M}(z_1,\ldots,z_m,t)$ denote a set of sensor responses:

$$\mathbf{M}(z_1,\ldots,z_m,t) = \left[h_1(\mathbf{x}(z_1,t))\ldots h_m(\mathbf{x}(z_m,t))\right]^T \tag{14}$$

The Gram determinant:

$$\gamma(z_1,\ldots,z_m) = \det\left[\int_0^T \mathbf{M}(z_1,\ldots,z_m,t)\mathbf{M}^T(z_1,\ldots,z_m,t)\,dt\right] \tag{15}$$

is maximized with respect to the sensor locations z_1, …, z_m in order to determine the best sensor configuration. This independence criterion is not meant to answer completely the problem of sensor location and system observability, but can be combined advantageously with simulation studies, process knowledge, experience and intuition, which remain important decision factors!

3. Optimal Estimation and Kalman Filtering

Optimal estimation techniques allow a minimum error estimate (according to some stated criterion of optimality) of the state to be obtained based on a dynamic model of the process, and knowledge of the statistics of process and measurement noises. The most common optimal estimation technique is the unbiased, minimum-error variance filter developed by Kalman for reconstructing the state of linear LPSs. In this section, the development and application of optimal estimation techniques to DPSs are briefly discussed. As mentioned in the introduction, the design of a state estimator for a DPS can proceed in either of two ways: early lumping or late lumping. The early lumping approach, which makes use of well-known techniques developed for LPSs, is first presented on the basis of examples (6) and (7). Then, attention is paid to the more general, late lumping approach.

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Bibliography

Ajinkya M.B., Ray W.H., Yu T.K., and J.H. Seinfeld. (1975). The application of an approximate non-linear filter to systems governed by coupled ordinary and partial differential equations. *International Journal of Systems Science*, **6**, 313–332. [This pioneering paper presents a general nonlinear DP filter and refers to several earlier works by the same authors that form the foundations of nonlinear filtering in DPSs.]

Bitzer M. and Zeitz M. (2002). Design of a nonlinear distributed parameter observer for a pressure swing adsorption plant. *Journal of Process Control* **12**, 33–543. [This paper discusses the application of a nonlinear DP observer to a two-bed pressure swing adsorption plant for oxygen production from air.]

Gilles E.D. (1973). *Systeme mit verteilten Parametern—Einführung in die Regelungstheorie (Distributed Parameter Systems: Introduction to Control Theory).* München-Wien: R. Oldenbourg Verlag. 186 pp. [This pioneering book introduces essential aspects and approaches to DPS modeling and control.]

Hua X., Mangold M., Kienle A., and Gilles E.D. (1998). State profile estimation of an authothermal periodic fixed-bed reactor. *Chemical Engineering Science* **53**, 47–58. [This paper presents the application of a nonlinear DP observer to a fixed-bed reactor used for combustion of volatile organic components in exhaust air streams emitted by polymer plants.]

Köhne M. (1977). *Zustandsbeobachter für Systeme mit verteilten Parametern: Theorie und Anwendung (State Observers for Distributed Parameter Systems: Theory and Application). Progress Report No. 8/26.* Düsseldorf, Germany: VDI-Verlag. 112 pp. [This research monograph explores the extensions of linear state estimation techniques to DPSs and discusses the application to a heat conduction system.]

Lappus G. and Schmidt G. (1990). Process monitoring and control of gas pipeline networks. *Real Time Microcomputer Control of Industrial Processes* (ed. S.G. Tzafestas and J.K. Pal), pp. 397–433. Dordrecht: Kluwer Academic Publishers. [This paper discusses the application of a nonlinear DP observer to an industrial high-pressure natural gas pipeline network.]

Mangold M., Lauschke G., Schaffner J., Zeitz M., and Gilles E.D. (1994). State and parameter estimation for adsorption columns by nonlinear distributed parameter state observers. *Journal of Process Control* **4**, 163–172. [This paper presents the application of a nonlinear DP observer to a fixed-bed adsorption column used for fluid phase separation of paraxylene and metaxylene using toluene.]

Marquardt W. and Auracher H. (1990). An observer-based solution of inverse heat conduction problems. *International Journal of Heat and Mass Transfer* **33**, 1545–1562. [This paper considers nonlinear DP observer to solve inverse heat conduction problems arising in the experimental study of wetting characteristics and heat transfer laws in forced convection transition boiling.]

Ray W.H. (1981). *Advanced Process Control.* New York: McGraw-Hill. 376 pp. [This widely used textbook presents essential aspects and approaches to DPS modeling, state estimation, and control, with a special emphasis on chemical engineering applications.]

Ruppel W. (1980). *Eine mathematische Beschreibung wandernder Reaktionszonen in Schüttschichten (Mathematical Description of Moving Reaction Zones in Fixed Beds).* Doctoral thesis, Department of Process Engineering, University of Stuttgart, Germany. [This PhD thesis presents the experimental application of a nonlinear DP observer to a tubular fixed-bed reactor in which an exothermic reaction takes place.]

Vande Wouwer A., Point N., Porteman S., and Remy M. (2000). An approach to the selection of optimal sensor locations in distributed parameter systems. *Journal of Process Control* **10**, 291–300. [This paper studies the problem of optimal sensor location in DPSs, with respect to system identification and state estimation.]

Vande Wouwer A., Zeitz M., Point N. and Remy M. (1993). On-line implementation of a nonlinear distributed observer for a multizone furnace: Comparative study with a nonlinear filter. *Control Engineering Practice* **1**, 947–955. [This paper presents the design and application of a nonlinear DP observer as well as a nonlinear DP filter to the reconstruction of the temperature profile in a furnace.]

Zeitz M. (1977). *Nichtlineare Beobachter für chemische Reaktore*n (Nonlinear Observers for Chemical Reactors). *Progress Report No. 8/27*. Düsseldorf, Germany: VDI-Verlag. 75 pp. [This research monograph is dedicated to the design of nonlinear state estimation techniques for lumped and distributed chemical processes.]

Biographical Sketches

Alain Vande Wouwer was born in Ixelles, Belgium, in 1966. He graduated in electrical engineering from the Faculté Polytechnique de Mons, Belgium, in 1988 and received the European doctorate degree in 1994 (Faculté Polytechnique de Mons–Stuttgart University). In 1994, he spent a postdoctoral period in the Mechanical Engineering Department at Laval University, Quebec, Canada. Presently he is Associate Professor in the Control Department of the Faculté Polytechnique de Mons. His research interests are in distributed parameter systems, parameter and state estimation, and bioprocess control.

Michael Zeitz was born in Frankfurt am Main, Germany, in 1940. He graduated in electrical engineering from Darmstadt Technical University, Germany, in 1967 and received the doctorate degree in 1973 and habilitation in 1977 from Stuttgart University, Germany. He became Professor of Simulation Engineering at Ruhr University in Bochum, Germany in 1977, and since 1979 he has been Professor of Control Engineering at Stuttgart University. He was Chairman (1994–1996) and Second Chairman (1993 and 1997) of the Department of Chemical Engineering and Engineering Cybernetics at Stuttgart University. He received the Landesforschungspreis (Research Prize) of Baden-Württemberg, Germany, in 1992, and the Honorary Doctorate of the Technical University in Donetsk, Ukraine in 1999. His research interests are in nonlinear control and observers, distributed parameter systems, and computer tools for modeling and simulation.