DISTRIBUTED PARAMETER SYSTEMS: AN OVERVIEW

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Contents

Summary

Control systems are ubiquitous in the modern world, where the instruments of our scientific and industrial society are applied to an increasingly wide range of processes. Such control intervention is undertaken with many different objectives in mind; e.g., “steering” the process to a desired state, minimizing the effects of various disturbances tending to move the system in undesirable directions, stabilizing systems which are inherently unstable or improving the stability properties of systems with weak stability characteristics, etc. While it is rarely possible, in a mathematical model, to account for all of the factors affecting the performance of a real world system, mathematical modeling of the system is, nevertheless, ordinarily essential for efficient and effective design and implementation of control procedures. In this chapter a particular class of mathematical control systems, often described in the literature as *distributed parameter systems* is described. We review the properties of these systems and compare them with those of other types of mathematical control systems. Additionally, we provide some indication how these distributed parameter systems function in the modeling of a variety of systems important in applications. Because the range and variety of theorems is very

great, each with its own set of specialized assumptions, we adopt a narrative approach to our account here rather than a "Theorem-Lemma-Proof" framework more suited to detailed discussion within a more limited context.

1. Introduction : Mathematical Control Systems

The subject of control of distributed parameter systems is vast; the available literature consists of literally thousands of articles on every conceivable aspect of what, is by, its very nature, a subject of great diversity. Any representative bibliography would, literally, fill all of the pages allotted to us for this chapter. In 1978 the author attempted a review of certain aspects of the subject as it had been developed to that time, now almost a quarter century ago. The bibliography, very incomplete even then, lists over one hundred contributions, including earlier reviews with even more extensive references. A comparable, but more recent, review has been provided by L.W.Markus. At the present writing there exists a wide variety of comprehensive texts treating various aspects of the subject with varying degree of completeness and mathematical sophistication. Even so, and quite understandably, none of these works attempts a complete treatment of the whole subject of distributed parameter systems control.

Inevitably, then, certain subjects are emphasized at the expense of others. In this chapter, for example, we do not discuss at all the very important question of *system identification* in the distributed parameter context-a subject on which literally hundreds of first rate books and articles have been written. We say very little about frequency domain approaches to distributed parameter systems, to which whole schools of academic and industrial researchers have addressed their efforts. An extensive literature has been devoted to the question of the existence, design etc., of state estimators and compensators, particularly finite dimensional compensators, in the context of infinite dimensional systems; we have little to say about this as well. We deal primarily with what we regard (subjectively, of course) as the "core" of the control theory of distributed parameter systems; the controllability and stabilizability theory of systems governed by partial differential and functional –differential equations.

A mathematical control system is a dynamical system involving *state variables, control variables, disturbance variables, measurement variables, measurement errors, and system parameters.* Complementing these are sets of dynamical equations serving to determine system evolution over specified intervals of time or regions of space, reference criteria, such as target states to be reached or trajectories to be tracked, etc. These are more possibilities than we can enumerate. We will begin our discussion with a brief recapitulation of finite dimensional systems with a view to contrasting these with infinite dimensional, or distributed parameter systems, which are the main subject of this chapter.

1.3 Finite Dimensional Systems

Here the state space is taken to be $\mathbf{E}^n$, the standard n-dimensional vector space; state vectors take the form

$$\mathbf{w} \equiv \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix},$$

the w_k constituting the *state components*. Each of the other variables and parameters would have a similar representation. If the process is continuous in time the dynamical equations would typically be ordinary differential equations involving the state, control, and disturbance variables:

$$\frac{d\mathbf{w}}{dt} = \mathbf{f}(\mathbf{w}, \mathbf{u}, \mathbf{v}).$$

In some contexts (for example, in the economic context with states corresponding to periodically reported economic quantities) the time variable might be taken to be discrete, so that we would have a *recursion,* or *difference* equations instead of a differential equation:

$$\mathbf{w}_{k+1} = \mathbf{f}(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}_k).$$

The most widely studied class of finite dimensional control systems is the class of systems for which the governing equation takes the form of a linear system of differential equations

$$\frac{d\mathbf{w}}{dt} = \mathbf{A}\mathbf{w} + \mathbf{B}\,\mathbf{u},$$

wherein the state vector $\mathbf{w} \in \mathcal{W} = \mathbf{E}^n$ and the control vector $\mathbf{u} \in \mathcal{U} = \mathbf{E}^m$ for some positive integers *n*, *m*; $\mathbf{A}$ and $\mathbf{B}$ are then $n \times n$ and $n \times m$ dimensional matrices, respectively. Such a system is *controllable* if and only if, given $T > 0$ and arbitrary initial and terminal states $\mathbf{w}_0, \mathbf{w}_1, \in \mathbf{E}^n$ there is a control function $\mathbf{u}(t) \in L^2(0,T;\mathbf{E}^m)$ (i.e., norm square integrable functions with values in $\mathbf{E}^m$)such that the solution of the indicated system of differential equations determined by $\mathbf{w}_0$ and $\mathbf{u}(t)$, i.e.,

$$\mathbf{w}(t) = e^{\mathbf{A}t}\mathbf{w}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau,$$

satisfies $\mathbf{w}(T) = \mathbf{w}_1$. As developed in the indicated references, this is equivalent to a number of other conditions. A purely algebraic necessary and sufficient condition for controllability is

$$rank[\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B} \ldots \mathbf{A}^{\nu-1}\mathbf{B}\ ] = n$$

for some non-negative integer $v \le n$. Another necessary and sufficient condition is that the so-called *controllability Grammian matrix*

$$\mathcal{G}_T = \int_0^T e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^* e^{\mathbf{A}^*(t-\tau)} d\tau \ ,$$

which is clearly self-adjoint and non-negative, should in fact be positive definite. The matrix $\mathcal{G}_T$ has the further important property that, assuming positive definiteness, and hence invertibility, the control of least norm "steering" $\mathbf{w}_0$ to $\mathbf{w}_1$ during the interval $[0,T]$ is given by

$$\tilde{\mathbf{u}}(\tau) = \mathbf{B}^* e^{\mathbf{A}^*(t-\tau)}\mathbf{z},\ \mathbf{z} = \mathcal{G}_T^{-1}\left(\mathbf{w}_1 - e^{\mathbf{A}T}\mathbf{w}_0\right) \in \mathbf{E}^n.$$

In linear system theory an important role is played by the *dual observed system*. In general a *linear observed system* takes the form

$$\frac{d\mathbf{z}}{dt} = \mathbf{C}\mathbf{z},\ \ \mathbf{y}(t) = \mathbf{O}\,\mathbf{z}(t) \quad .$$

The first being the governing differential equation for the system, while the second indicates a linear *output*, *measurement* or *observation* relation, giving the observation $\mathbf{y}(t)$ in terms of the state trajectory $\mathbf{z}(t)$. When the matrix $\mathbf{C}$ coincides with $-\mathbf{A}^*$ and matrix $\mathbf{O}$ coincides with $\mathbf{B}^*$ the resulting linear observed system

$$\frac{d\mathbf{z}}{dt} = -\mathbf{A}^*\mathbf{z},\ \ \mathbf{y}(t) = \mathbf{B}^*\mathbf{z}(t)$$

is the *dual linear observed system* for the (primal) control system originally introduced. A general linear observed system $\frac{d\mathbf{z}}{dt} = \mathbf{C}\mathbf{z},\ \mathbf{y}(t) = \mathbf{O}\mathbf{z}(t)$ is *observable* if the output $\mathbf{y}(t), t \in [0,T]$, determines the initial state $\mathbf{z}(0) = \mathbf{z}_0$. That is the case if and only if the *observability Grammian matrix*

$$\mathcal{H}_T = \int_0^T e^{\mathbf{C}^* t}\mathbf{O}^*\mathbf{O} e^{\mathbf{C}^t} dt.$$

is positive definite, in which event

$$\mathbf{z}_0 = \mathcal{H}_T^{-1} \int_0^T e^{\mathbf{C}^* t}\mathbf{O}^*(t) dt.$$

When $\mathbf{C} = -\mathbf{A}^*$ and $\mathbf{O} = \mathbf{B}^*$ it is easy to see that positive definiteness of $\mathcal{H}_T$ holds just in case $\mathcal{G}_T$ is a positive definite as well and we conclude that the primal linear control system is controllable if and the dual linear system is observable. This may be considered to be fundamental principle of the theory of linear control systems.

1.4 Function Spaces as System Spaces

In the case of finite dimensional control systems we typically have

$$\mathbf{w} \in \mathbf{R}^n,\ \mathbf{u} \in \mathbf{R}^m,\ \mathbf{v} \in \mathbf{R}^p,\ \mathbf{y} \in \mathbf{R}^q,$$

etc, the finite dimensional spaces $\mathbf{R}^n, \mathbf{R}^m, \mathbf{R}^p, \mathbf{R}^q$, etc., serving, respectively, as the *state space*, *control space, disturbance space, measurement space*, etc.(henceforth we will refer to these as the *system spaces* to avoid excessive repetition). Little more needs to be said about these in the finite dimensional case; any questions of regularity (smoothness) of system trajectories arise only as questions of regularity with respect to the time variable. In the case of distributed parameter systems with spatial coordinate variables $\mathbf{x} \in \mathcal{R} \subset \mathbf{R}^v$ and system variables and /or parameters expressed as functions of $\mathbf{x}$ and *t*, the situation is markedly different and we have to provide a much more detailed specification of the system spaces in order to admit an adequate treatment of system properties and behavior.

Let us begin by supposing the spatial region of interest to be closed region $\mathcal{R} \subset \mathbf{R}^v$. (In some cases this spatial region might vary with *t* but this is relatively uncommon and we will not let that possibility complicate our discussion here). The region $\mathcal{R}$ might be finite or infinite in extent, but we will suppose that, in all circumstances, its boundary is piecewise smooth, being the union of smooth *v*-1-dimensional manifolds, most typically points in the case $v = 1$, curves in the case $v = 2$ and two-dimensional surfaces in the case $v = 3$.

The space $\mathbf{C}(\mathcal{R}) \equiv \mathbf{C}^0(\mathcal{R})$ consists of all functions $\mathbf{f}(\mathbf{x})$ defined and continuous for $\mathbf{x} \in \mathcal{R} \subset \mathbf{R}^v$. If it is necessary to specify the *range dimensions,* μ, of $\mathbf{f}$, we will write $\mathbf{C}_\mu(\mathcal{R})$ or $\mathbf{C}^0_\mu(\mathcal{R})$. This is a *complete normed linear space*, or *Banach space,* with the norm

$$\|\mathbf{f}\|_0 = \sup_{\mathbf{x} \in \mathcal{R}} \left\{ \|\mathbf{f}(\mathbf{x})\|_\mu \right\},$$

where $\|\mathbf{f}(\mathbf{x})\|_\mu$ is an appropriate norm, e.g., the standard Euclidean norm, for vectors in $\mathbf{R}^\mu$. Spaces whose elements have greater smoothness than general elements of $\mathbf{C}^0(\mathcal{R})$ are the spaces $\mathbf{C}^k(\mathcal{R})$, or $\mathbf{C}^k_\mu(\mathcal{R})$, consisting of functions $\mathbf{f}(\mathbf{x})$ such that all partial derivatives of $\mathbf{f}(\mathbf{x})$ of order $j, 0 \le j \le k,$ lie in $\mathbf{C}^0(\mathcal{R})$ (in fact, they must then lie in $\mathbf{C}^{k-j}(\mathcal{R})$). These also are Banach spaces with norm $\|\mathbf{f}\|_k$ taken to be the largest norm in $\mathbf{C}^0(\mathcal{R})$ of any partial derivative of $\mathbf{f}(\mathbf{x})$ of order $j, 0 \le j \le k$.

Hilbert spaces, i.e., complete *inner-product* spaces play a very strong role in the study of distributed parameter systems, primarily because the inner product structure relates in a very natural way to a variety of kinetic and potential energy forms. These energy forms, combined with energy conservation or dissipation properties, are often indispensable in studying existence and regularity properties and establishing stability properties of distributed systems. With $\mathcal{R}$ a region as described above, the *Sobolev spaces*, $H_\mu^m(\mathcal{R})$, consist of certain μ-dimensional functions **f** defined on $\mathcal{R}$, possessing partial derivatives of all orders less than or equal to *m*; these partial derivatives may exist in a *distributional*, rather than the classical sense. For ease of exposition we suppose that **Df** represents a partial derivative operator on **f:**

$$\mathbf{Df} \equiv \frac{\partial^p \mathbf{f}}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_\mu^{p_\mu}} \ ,$$

where p_k are non-negative integers whose sum is the non-negative integer *p*. For any such differential operator the order of **D** is $|\mathbf{D}| = p$. The function $\mathbf{f} \in H_\mu^m(\mathcal{R})$ are precisely those for which the *m*-th order *Sobolev norm* is finite, i.e.,

$$\|\mathbf{f}\| \equiv \sqrt{\sum_{|\mathrm{D}|\le m} \int_{\mathcal{R}} \|\mathbf{Df}(\mathbf{x})\|^2 \, d\mathbf{x}} < \infty .$$

Here the norm $\|\mathbf{Df}(\mathbf{x})\|$ is simply the Euclidean norm in $\mathbf{R}^\mu$ of the vector $\mathbf{Df}(\mathbf{x})$ and $d\mathbf{x}$ indicates the standard measure in $\mathbf{R}^\mu$. When *m* = 0 the space $H_\mu^m(\mathcal{R})$ reduces to the familiar Lebesgue space $H_\mu^0(\mathcal{R}) = L_\mu^2(\mathcal{R})$. (In the sequel we will suppress the dimensional subscript μ, unless it is necessary to specify or refer to that dimension, and just write $H^m(\mathcal{R})$.) The *Sobolev inner product* in $H^m(\mathcal{R})$ is the bilinear (sesquilinear) if the functions involved have complex values) form

$$\langle f, g \rangle = \sum_{|\mathrm{D}|\le m} \int_{\mathcal{R}} \langle \mathbf{Df}(\mathbf{x}), \mathbf{Dg}(\mathbf{x}) \rangle \, d\mathbf{x},$$

where $\langle \mathbf{Df}(\mathbf{x}), \mathbf{Dg}(\mathbf{x}) \rangle$ in the integrand is just the standard inner ("dot") product in $\mathbf{R}^\mu$, again with the usual modifications of the values involved are complex. This form possesses the standard inner product properties and admits the identity $\|\mathbf{f}\| = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle}$.

Spaces with very similar properties, often closely connected with significant applications, can be obtained by replacing the sum $\sum_{|\mathbf{D}|\le m} \|\mathbf{Df}(\mathbf{x})\|^2$ with particular non-negative quadratic forms in the partial derivatives. For example, in two dimensional linear elasticity with displacement state vector $\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ defined in a region $\mathcal{R}$, the relevant (potential energy) quadratic form for a uniform material is

$$\int_{\mathcal{R}} \left(\frac{\lambda_1}{2} \left(\frac{\partial u^2}{\partial x} + \frac{\partial v^2}{\partial y} \right) + \frac{\lambda_1}{4} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \frac{\lambda_2}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 \right) dx\, dy,$$

where λ_1 and λ_2 are the Lamé constants. The corresponding inner product is the *energy inner product* and Hilbert space thus obtained is the *potential energy space* for two-dimensional linear elasticity.

There are many other examples of function spaces important for particular systems; e.g., many specialized Hilbert and Banach spaces arise in connection with functional differential equations of time delay type. Space does not permit us to attempt even a representative selection of these for description here. The necessity, and desirability, of choosing the right state space for a distributed parameter system and then demonstrating that the relevant time trajectories of the system remain within the state space is one of the most challenging, and sometimes frustrating, aspects of the field.

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