MATHEMATICS: CONCEPTS AND FOUNDATIONS

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CONTENTS

VOLUME I

A View of Mathematics 1
Alain Connes, Institute des Hautes Etudes Scientifiques, France

1. The Unity of Mathematics
2. The concept of Space
   2.1. Projective Geometry
   2.2. The Angel of Geometry and the Devil of Algebra
   2.3. Non-Euclidean Geometry
   2.4. Symmetries
   2.5. Line element and Riemannian geometry
   2.6. Noncommutative Geometry
   2.7. Grothendieck’s Motives
   2.8. Topos theory
3. Fundamental Tools
   3.1. Positivity
   3.2. Cohomology
   3.3. Calculus
   3.4. Trace and Index Formulas
   3.5. Abelian Categories
   3.6. Symmetries
4. The input from Quantum Field Theory
   4.1. The Standard Model
   4.2. Renormalization
   4.3. Symmetries

Mathematics through Millenia 43
Vagn Lundsgaard Hansen, Department of Mathematics, Technical University of Denmark, Denmark

1. Introduction
2. The dawn of mathematics
   2.1. Egyptian Mathematics
   2.2. Mesopotamian Mathematics
   2.3. Mayan Mathematics
3. The Greek heritage in mathematics
   3.1. Geometry
   3.2. Number Theory
4. The golden period of the Hindus and the Arabs in mathematics
   4.1. Hindu Mathematics
   4.2. Islamic Mathematics
   4.3. Mathematics in Europe in the Middle Ages
5. Mathematics in China
   5.1. Ancient Chinese Mathematics
   5.2. The “Nine Chapters on the Mathematical Art”
   5.3. In the Shadows of the Great Masters
   5.4. A Golden Century for Mathematics in China
6. European mathematics in the Renaissance
   6.1. The Solution of Cubic Equations
   6.2. Mathematics inspired by Applications
7. Mathematics and the scientific revolution
   7.1. Analytic Geometry
   7.2. Calculus gets off the Ground
   7.3. Other Mathematical Discoveries from the Seventeenth Century
8. The tools of calculus are developed and consolidated
   8.1. The Birth of Mathematical Analysis
   8.2. Further Remarks on Mathematics in the Eighteenth Century
9. Abstract mathematical structures emerge
   9.1. New Algebraic Structures
   9.2. Groundbreaking New Discoveries in Geometry
   9.3. Rigor in Analysis
   9.4. Further Developments in the Nineteenth Century
10. Mathematics in the twentieth century
    10.1. Problems in the Foundations of Set Theory
    10.2. Tendencies in Twentieth Century Mathematics
    10.3. Highlights from Twentieth Century Mathematics
11. Mathematics forever
1. Matrices, Vectors and their Basic Operations
   1.1. Matrices
   1.2. Vectors
   1.3. Addition and Scalar Multiplication of Matrices
   1.4. Multiplication of Matrices
2. Determinants
   2.1. Square Matrices
   2.2. Determinants
   2.3. Cofactors and the Inverse Matrix
3. Systems of Linear Equations
   3.1. Linear Equations
   3.2. Cramer’s Rule
   3.3. Eigenvalues of a Complex Square Matrix
   3.4. Jordan Canonical Form
4. Symmetric Matrices and Quadratic Forms
   4.1. Real Symmetric Matrices and Orthogonal Matrices
   4.2. Hermitian Symmetric Matrices and Unitary Matrices
5. Vector Spaces and Linear Algebra
   5.1. Vector spaces
   5.2. Subspaces
   5.3. Direct Sum of Vector Spaces
   5.4. Linear Maps
   5.5. Change of Bases
   5.6. Properties of Linear Maps
   5.7. A System of Linear Equations Revisited
   5.8. Quotient Vector Spaces
   5.9. Dual Spaces
   5.10. Tensor Product of Vector Spaces
   5.11. Symmetric Product of a Vector Space
   5.12. Exterior Product of a Vector Space

<table>
<thead>
<tr>
<th>Groups and Applications</th>
<th>152</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tadao Oda, Tohoku University, Japan</td>
<td></td>
</tr>
<tr>
<td>1. Groups</td>
<td></td>
</tr>
<tr>
<td>2. Commutative Groups</td>
<td></td>
</tr>
<tr>
<td>3. Examples</td>
<td></td>
</tr>
<tr>
<td>4. Subgroups</td>
<td></td>
</tr>
<tr>
<td>5. Homomorphisms</td>
<td></td>
</tr>
<tr>
<td>6. Quotient Groups</td>
<td></td>
</tr>
<tr>
<td>7. Homomorphism and Isomorphism Theorems</td>
<td></td>
</tr>
<tr>
<td>8. Cyclic Groups</td>
<td></td>
</tr>
<tr>
<td>9. Direct Products</td>
<td></td>
</tr>
<tr>
<td>10. Finitely Generated Abelian Groups</td>
<td></td>
</tr>
<tr>
<td>11. Group Actions and Symmetry</td>
<td></td>
</tr>
<tr>
<td>12. Solvable Groups</td>
<td></td>
</tr>
<tr>
<td>13. Representations of Finite Groups</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rings and Modules</th>
<th>179</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tadao Oda, Tohoku University, Japan</td>
<td></td>
</tr>
<tr>
<td>1. Definition of Rings</td>
<td></td>
</tr>
<tr>
<td>2. Basic Properties and Examples</td>
<td></td>
</tr>
<tr>
<td>3. Noetherian Rings</td>
<td></td>
</tr>
<tr>
<td>4. Completion</td>
<td></td>
</tr>
<tr>
<td>5. Localization and Local Rings</td>
<td></td>
</tr>
</tbody>
</table>
6. Modules
7. Integral Extensions

**Fields and Algebraic Equations**
Tadao Oda, *Tohoku University, Japan*

1. Basic Properties and Examples of Fields
2. Algebraic Equations
3. Algebraic Extensions
4. Separability
5. Galois Theory
6. Finite Fields
7. Cyclotomic Extensions
8. Kummer Extensions
9. Solvability
10. Ruler and Compass Constructions

**Number Theory and Applications**
Katsuya Miyake, *The Graduate School of Science and Engineering, Waseda University, Japan*

1. The Additive Structure of Natural Numbers
   1.1. The Well-Ordered Structure and the Principle of Mathematical Induction
   1.2. Triangular Numbers and Square Numbers
2. The Multiplicative Structure of Natural Numbers
   2.1. Prime Numbers
   2.2. Infinitude of Prime Numbers and Euler Product
   2.3. Euclidean Algorithm and the Greatest Common Divisors
   2.4. Dirichlet’s Prime Number Theorem on Arithmetic Progressions
3. The Ring of Integers
   3.1. The Ring of Integers
   3.2. Linear Equations in Integers and Divisibility
   3.3. Multiplicative Structure of the Integral Solutions of Pell’s Equations
   3.4. Multiplicative Structure on Binary Quadratic Equations
4. Congruence
   4.1. Congruence Relation and Residue Rings
   4.2. Euler’s Phi Function
   4.3. Chinese Remainder Theorem
   4.4. Linear Congruence Equations
   4.5. Quadratic Congruence Equations and Quadratic Residues
   4.6. The Reciprocity Law of Quadratic Residues
   4.7. The Multiplicative Group of a Finite Field and Primitive Roots modulo $p$
   4.8. Caesar’s Cipher in Cryptography and Congruence
   4.9. Public Key Cryptology
5. Analytic Methods in Number Theory
   5.1. Counting Prime Numbers
   5.2. Densities of some Sets of Prime Numbers
   5.3. The Riemann Zeta Function and the Riemann Hypothesis
   5.4. Dirichlet Characters and Dirichlet’s $L$-functions
6. Arithmetic of Quadratic Fields
   6.1. Quadratic Fields and the Rings of Integers
   6.2. Ideals and the Fundamental Theorem of Arithmetic in a Quadratic Field
   6.3. Units of Quadratic Fields and Pell’s Equations
   6.4. Ideal Class Groups and Class Numbers
7. Cyclotomic Fields
   7.1. Algebraic Bases of Cyclotomic Fields
   7.2. Arithmetic Bases of Cyclotomic Fields
7.3. Kronecker-Weber Theorem on Abelian Polynomials over the Rational Number Field

8. Comments on Kronecker’s Dream in his Youth and Class Field Theory
   8.1. Kronecker’s Dream in his Youth
   8.2. The Ideal Class Group of an imaginary Quadratic Field and Automorphism Classes of Elliptic Function Fields with Complex Multiplication.

Algebraic Geometry and Applications
Tadao Oda, Tohoku University, Japan

1. Affine Algebraic Varieties
2. Projective Algebraic Varieties
3. Sheaves and General Algebraic Varieties
4. Properties of Algebraic Varieties
5. Divisors
6. Algebraic Geometry over Algebraically Closed Fields
7. Schemes
8. Applications

Basic Notions of Geometry and Euclidean Geometry
Tetsuya Ozawa, Department of Mathematics, Meijo University, Japan

1. Introduction
2. Basic Notions
   2.1. Metric Space
   2.2. Transformation Group
   2.3. Lie Group
3. Euclidean Space
   3.1. Euclidean Vector Space
   3.2. Euclidean Space
   3.3. Equations of Plane and Sphere
   3.4. Triangle and Plane Trigonometry
4. Euclidean Group
   4.1. Translation and Rotation
   4.2. Categorization of Isometries
5. Conic Sections
   5.1. Binary Quadratic Equation
   5.2. Focus, Eccentricity and Directrix
   5.3. Confocal Conic Sections
   5.4. Inverse Square Central Force
6. Discrete Groups of Isometries
   6.1. Finite Subgroups of Isom (E^n) and Polyhedra
   6.2. Space Group

Affine Geometry, Projective Geometry, and Non-Euclidean Geometry
Takeshi Sasaki, Department of Mathematics, Kobe University, Japan

1. Affine Geometry
   1.1. Affine Space
   1.2. Affine Lines
   1.3. Affine transformations
   1.4. Affine Collinearity
   1.5. Conic Sections
2. Projective Geometry
   2.1. Perspective
   2.2. Projective Plane

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2.3. Projective Transformations
2.4. Projective Collinearity
2.5. Conics

3. Geometries and Groups
3.1. Transformation Groups
3.2. Erlangen Program

4. Non-Euclidean Geometry
4.1. Elliptic Geometry
4.2. Hyperbolic Geometry
4.3. Poincaré Model
4.4. Riemannian Geometry

Differential Geometry
Takashi Sakai, Department of Applied Mathematics, Okayama University of Science, Japan

1. Curves in Euclidean Plane and Euclidean Space
2. Surfaces in Euclidean Space
3. Differentiable Manifolds
4. Tensor Fields and Differential Forms
5. Riemannian Manifolds
6. Geometric Structures on Manifolds
7. Variational Methods and PDE

Topology
Takashi Tsuboi, Graduate School of Mathematical Sciences, University of Tokyo, Japan

1. Introduction
2. Convergence of sequences, continuity of maps, general topology
   2.1. Metric Spaces and the Convergence of Sequences
   2.2. Abstract Topology on Sets
   2.3. Separation Axioms and Countability Axioms
   2.4. Compactness
3. Connectedness and homotopy theory
   3.1. Connectedness and Homotopy
   3.2. Homotopy Groups
   3.3. Homotopy Exact Sequence
4. Simplicial complexes and homology theory
   4.1. Simplicial Complexes
   4.2. Chain Complexes and Homology Groups
   4.3. Singular Simplicial Complex and Singular Homology Groups
   4.4. Cochain Complexes and Cohomology Groups
5. Applications for manifold theory
   5.1. Poincaré Duality Theorem
   5.2. Poincaré-Hopf Theorem
   5.3. Morse Theory
   5.4. Homotopy Type and Homeomorphism Type
   5.5. Intersection Numbers and Linking Numbers
   5.6. Knot Theory

Complex Analytic Geometry
Tatsuo SUWA, Department of Information Engineering, Niigata University, Japan

1. Analytic functions of one complex variable
2. Analytic functions of several complex variables
3. Germs of holomorphic functions
4. Complex manifolds and analytic varieties  
5. Germs of varieties  
6. Vector bundles  
7. Vector fields and differential forms  
8. Chern classes of complex vector bundles  
9. Divisors  
10. Complete intersections and local complete intersections  
11. Grothendieck residues  
12. Residues at an isolated zero  
13. Examples  
14. Sheaves and cohomology  
15. de Rham and Dolbeault theorems  
16. Poincaré and Kodaira-Serre dualities  
17. Riemann-Roch theorem  

Index  

About EOLSS
<table>
<thead>
<tr>
<th>Mathematics: Concepts, and Foundations</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.2. C’ – Functions</td>
</tr>
<tr>
<td>6.3. Total Differential</td>
</tr>
<tr>
<td>6.4. Derivatives of Composite Functions.</td>
</tr>
<tr>
<td>6.5. Taylor’s Formula for Functions of Several Variables</td>
</tr>
<tr>
<td>6.6. Extrema of Functions of Several Variables</td>
</tr>
<tr>
<td>7. Multiple Integrals</td>
</tr>
<tr>
<td>7.1. Riemann Integrals</td>
</tr>
<tr>
<td>7.2. The Iterated Integral</td>
</tr>
<tr>
<td>7.3. Change of Variables in Multiple Integrals</td>
</tr>
</tbody>
</table>

**Complex Analysis**

Sin Hitotumatu, *Kyoto University, Japan*

1. Complex numbers
2. Holomorphic functions
   2.1. Conditions for Holomorphic Functions
   2.2. Examples of Holomorphic Functions
   2.3. Zero and an Isolated Singularity
   2.4. Analytic Functions and Analytic Continuation
3. Residue and residue calculus
4. Analytic functions of several complex variables
5. Brief history

**Measure and Probability**

Hisoa Watanabe, *Professor emeritus, Kyushu University, Fukuoka, Japan*

1. Introduction
2. Measure
   2.1. Fields of Sets
   2.2. Lebesgue Measure
   2.3. Measures
   2.4. Measurable Functions
   2.5. Integral
   2.6. Product Measures
   2.7. Relations between two Measures
   2.8. Signed Measures
   2.9. Radon Measures
   2.10. Haar Measures
3. Probability
   3.1. Basic Definitions and Results
   3.2. Sum of Independent Random Variables- Infinite Divisible Distributions
   3.3. Conditional Expectation and Martingale
      3.3.1. Conditional Expectation
      3.3.2. Martingale
   3.4. Stationary Process- Ergodic Theory
      3.4.1. Discrete Parameter
      3.4.2. Continuous Parameter
      3.4.3. Stationary Gaussian Processes
   3.5. Markov Processes
      3.5.1. Heat Equation and Corresponding Markov Processes.
      3.5.2. Markov Chains
   3.6. Stochastic Dynamical System- Itô Calculus

**Functional Analysis and Function Spaces**

Mikihiro Hayashi, *Department of Mathematics, Hokkaido University, Japan*
1. Introduction
2. Function Spaces and Some Examples
3. Basic Concepts in Functional Analysis
   3.1. Normed spaces and Banach Spaces
   3.2. Hilbert Spaces
   3.3. Bounded Linear operators
   3.4. Applications of Bair's Category Theorem
   3.5. The Dual Space of a Banach Space
   3.6. The Duality of Hilbert Spaces
4. Some Advanced Concepts in Functional Analysis
   4.1. Topological Vector Spaces
   4.2. The Weak Topology and the Weak * Topology
   4.3. Locally Convex Spaces
   4.4. Banach Algebras
5. Miscellaneous Function Spaces
   5.1. Spaces of Continuous Functions
   5.2. Spaces of Measurable Functions
   5.3. Spaces of Differentiable Functions
   5.4. Spaces of Holomorphic Functions

Numerical Analysis and Computation
Yasuhiko Ikebe, Meisei University, Information Science Research Center, Meisei University, Hino City 191-8506, Japan
1. Linear Systems of Equations
2. An Example
3. Condition Number
4. Norms and Vector Spaces
5. Application to Error Analysis
6. Stable Algorithms and Stable Problems
7. Application to Numerical Solution of Linear Systems
8. Iterative Methods
9. Eigenvalue Problems
10. Singular Value Decomposition
11. Software and Remarks

Infinite Analysis
Tetsuji Miwa, Department of Mathematics, Kyoto University, Kitashirakawa Oiwakecho, Sakyo, Kyoto, Japan
1. Introduction
2. Ising Model and Monodromy Preserving Deformation
   2.1. Two-dimensional Ising Model and Onsager’s Result
   2.2. Transfer Matrix
   2.3. Harmonic Oscillator
   2.4. Clifford Algebra and Clifford Group
   2.5. Free Fermions and Creation/Annihilation Operators
   2.6. Magnetization and Scaled Two-point Correlation Function
   2.7. Ising Field Theory and Monodromy Preserving Deformation
3. Soliton Equations and Vertex Operators
   3.1. Bosonic Fock spaces and Vertex Operators
   3.2. $\tau$ Functions of KP and KdV Hierarchies
   3.3. $GL(\infty)$ – Orbit and $\tau$ Function
4. Conformal Coinvariants and Vertex Operators
   4.1. Affine Lie Algebra $\widehat{sl}_2$ and Vertex Operator Representations
4.2. Sugawara Construction and the Level $k$ Vertex Operators
4.3. Conformal Blocks and Coinvariants
5. XXZ Model and Quantum Vertex Operators
   5.1. Quantum Hamiltonian and Commuting Transfer Matrix
   5.2. Quantum Affine Algebra $\tilde{U}_q(\mathfrak{sl}_2)$ and $R$ Matrix
   5.3. Level One Modules and Space of States
   5.4. Method of Corner Transfer Matrix and Vacuum States
   5.5. Quantum Vertex Operator and Diagonalization of Transfer Matrix
6. Form Factor Bootstrap Approach in Sine-Gordon Model
   6.1. $S$-Matrix and Form Factor Axioms
   6.2. Hypergeometric Integrals and $\infty$-cycles.
   6.3. Level Zero Action Revisited

Fourier Analysis and Integral Transforms
Satoru Igari, Emeritus Professor of Tohoku University, Japan.

1. Introduction
2. Fourier series
   2.1. Definition
   2.2. Convolution and Fourier Series
   2.3. Pointwise Convergence of Fourier Series
   2.4. Norm Convergence of Fourier Series
   2.5. Analytic Functions in the Unit Disk
   2.6. Orthogonal Function Expansions
      2.6.1. Orthogonal Systems
      2.6.2. Examples of Orthogonal Systems
3. Wavelet expansion
   3.1. Multiresolution Analysis.
   3.2. Examples of Wavelets
4. Fourier transforms
   4.1. Fourier Transform in One Variable
      4.1.1. Definition and Inversion Formula
      4.1.2. Examples
      4.1.3. Convergence of Fourier Integrals
      4.1.4. Poisson Summation Formula
   4.2. Fourier Transform and Analytic Functions
      4.2.1. Hardy Space.
      4.2.2. Real Method in Hardy Spaces.
   4.3. Fourier Transform in Several Variables
      4.3.1. Definition and Examples
      4.3.2. Some Fundamental Properties
      4.3.3. Fourier Transform in the Spaces $L^2$ and $S$
5. Fourier analysis on locally compact Abelian groups
6. Finite Fourier Transform
   6.1. Finite Fourier Transform
   6.2. Fast Fourier Transform
7. Integral transforms
   7.1. Mellin Transform
   7.2. Hankel Transform
   7.3. Laplace Transform
   7.4. Wavelet Transform

Operator Theory and Operator Algebra
Hideki Kosaki, Graduate School of Mathematics, Kyushu University, Japan

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1. Hilbert space
2. Bounded linear operator
   2.1. Compact Operator
   2.2. Miscellaneous Operators
   2.3. Polar Decomposition and Spectral Decomposition
   2.4. Spectrum
3. Operator theory
   3.1. Dilation Theory
   3.2. Generalization of Normality
   3.3. Toeplitz Operator
   3.4. Operator Inequalities
4. Operator algebra
   4.1. C*-algebra
      4.1.1. Type I C*-algebra
      4.1.2. Nuclear C*-algebra
      4.1.3. Operator Algebra K-theory
      4.1.4. Purely Infinite C*-algebra
   4.2. von Neumann Algebra
      4.2.1. Basic Theory
      4.2.2. Modular Theory and Structure of Type III Factors
      4.2.3. Classification of AFD Factors
      4.2.4. Index Theory
      4.2.5. Free Probability Theory

Formal Logic

Yiannis N. Moschovakis, Department of Mathematics, University of California, Los Angeles, CA, USA and University of Athens, Greece

1. Cantor’s Set Theory
   1.1. Equinumerosity; Countable and Uncountable Sets
   1.2. Cardinal Arithmetic
   1.3. Transfinite Induction and Recursion
   1.4. The Situation in 1900
2. The Birth of First Order Logic
   2.1. Frege’s Logicism Program
   2.2. The First Order Language of Sets
   2.3. Logical Deduction
   2.4. Frege’s Set Theory
   2.5. Frege’s Definition of Number
3. The Paradoxes
   3.1. Intuitionism
   3.2. Type Theory
   3.3. Hilbert’s Program
4. Axiomatic Set Theory
   4.1. The Zermelo-Fraenkel Axioms
   4.2. The Axiom of Choice
   4.3. Von Neumann Ordinals and Cardinals
   4.4. The Cumulative Hierarchy of Sets
   4.5. The Emancipation of Logic from Set Theory
5. Mathematical Logic
   5.1. Propositional Logic, PL
   5.2. The Syntax of First Order Logic, FOL
   5.3. FOL-Semantics
   5.4. First Order (Elementary) Definability
   5.5. Model Theory and Non-Standard Models
   5.6. FOL-Deduction
   5.7. Soundness and Completeness of FOL
5.8. Second Order Logic, FOL
5.9. The Typed λ-Calculus, L
6. G"odel’s First Incompleteness Theorem
   6.1. The Incompleteness of Peano Arithmetic
   6.2. Outline of G"odel’s Proof
7. Computability and Unsolvability
   7.1. Turing Machines
   7.2. The Church-Turing Thesis
   7.3. Unsolvable Problems and Undecidable Theories
   7.4. G"odel’s Second Incompleteness Theorem
   7.5. The Arithmetical and Analytical Hierarchies
8. Recursion and Computation
   8.1. Recursive Programs
   8.2. Programming Languages

---

Model Theory
H. Jerome Keisler, Department of Mathematics, University of Wisconsin, Madison Wisconsin U.S.A
1. Introduction
2. Classical Model Theory
   2.1. Constructing Models
   2.2. Preservation and Elimination
   2.3. Special Classes of Models
3. Models of Tame Theories
   3.1. ω-stable Theories
   3.2. Stable and Simple Theories
   3.3. o-minimal Theories
4. Beyond First Order Logic
   4.1. Infinitary Logic with Finite Quantifiers
   4.2. Logic with Cardinality Quantifiers
5. Model Theory for Mathematical Structures
   5.1. Topological Structures
   5.2. Functional Analysis
   5.3. Adapted Probability Logic

---

Proof Theory and Constructive Mathematics
Anne S. Troelstra, ILLC, University van Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, Netherlands
1. Introduction
   1.1. Constructivism
   1.2. Proof Theory
2. Intuitionistic Logic, I
   2.1. The BHK-interpretation
   2.2. Natural Deduction; Formulas as Types
   2.3. The Hilbert-type Systems Hi and Hc
   2.4. Metamathematics of I and its Relation to classical logic C
3. Semantics of Intuitionistic Logic
   3.1. I-completeness
   3.2. Kripke semantics
   3.3. Topological and Algebraic Semantics
4. Intuitionistic (Heyting) Arithmetic, HA
   4.1. Realizability
   4.2. Characterization of Realizability
5. Constructive Mathematics
   5.1. Bishop’s Constructive Mathematics (BCM)
5.2. Constructive Recursive Mathematics (CRM)
5.3. Intuitionism (INT)
5.4. Lawless Sequences and I-validity
5.5. Comparison of BCM, CRM and INT
6. Proof Theory of First-order Logic
   6.1. The Gentzen systems Gc and Gi
   6.2. Cut Elimination
   6.3. Natural Deduction and Normalization
   6.4. The Tait-Calculus
7. Proof Theory of Mathematical Theories
   7.1. The language
   7.2. Order types
   7.3. Truth-Complexity
   7.4. The Proof-theoretic Ordinal of a Theory
   7.5. The Method

Computability and Complexity

Martin Davis, Professor Emeritus, Courant Institute, New York University, and Visiting Scholar, Mathematics Department, University of California, Berkeley, USA

1. Introduction
2. Recursive and Recursively Enumerable Sets
   2.1. m-Complete Sets, Creative Sets, and Simple Sets
   2.2. Algorithmic View of Gödel Incompleteness
3. Unsolvable Problems
   3.1. The Word Problem for Semigroups
   3.2. The Word Problem for Groups
4. Hilbert’s 10th Problem
   4.1. Applications and Extensions of MRDP
5. Classifying Unsolvable Problems.
   5.1. Degrees of Unsolvability
   5.2. The Arithmetical Hierarchy
   5.3. R.e. Degrees
6. Complexity
   6.1. Abstract Complexity Theory
   6.2. Polynomial-time Computability

Index

About EOLSS

VOLUME III

Set Theory

John R. Steel, Department of Mathematics, University of California, Berkeley, CA. USA

1. Introduction
2. Some Elementary Tools
   2.1. Ordinals
   2.2. The Wellordering Theorem
   2.3. The Cumulative Hierarchy; Proper Classes
   2.4. Cardinals
   2.5. Cofinality, Inaccessibility, and König’s Theorem
   2.6. Club and Stationary Sets
2.7. Trees
2.8. Transitive Models and the Levy Hierarchy
2.9. Large Cardinals and the Consistency-Strength Hierarchy
3. Constructible Sets
  3.1. Gödel’s Work on \( L \)
  3.2. Suslin Trees \( \Diamond \) and \( \Box \)
  3.3. Canonical Inner Models Larger Than \( L \)
4. Forcing
  4.1. The Basics of Forcing
  4.2. \( \neg \text{CH} \) via Adding Cohen Reals
  4.3. Easton’s theorem
  4.4. The Singular Cardinals Problem
  4.5. A model where the Axiom of Choice fails
  4.6. Cardinal Collapsing and Solovay’s Model
  4.7. Suslin’s Hypothesis and Martin’s Axiom
  4.8. Martin’s Maximum
5. Descriptive Set Theory
  5.1. Gödel’s Program
  5.2. Classical Descriptive Set Theory
  5.3. Determinacy
  5.4. Large Cardinals and Determinacy
  5.5. Generic Absoluteness and CH
6. Other Topics

**Logic and Computer Science**

Phokion G. Kolaitis, *Computer Science Department, University of California, Santa Cruz, CA 95064, USA*

1. Introduction
2. Complexity Classes and the \( P = \) NP problem
3. Propositional Logic and Complexity Classes
4. The Complexity of First-Order Logic and Richer Logics
   4.1. The Complexity of First-Order Logic
   4.2. The Complexity of Existential Second-Order Logic
   4.3. Fagin’s Theorem and Descriptive Complexity
   4.4. Least Fixed-Point Logic and Polynomial-Time
   4.5. Partial Fixed-Point Logic and Polynomial Space
5. Finite Model Theory
   5.1. Classical Model Theory in the Finite
   5.2. Ehrenfeucht-Fraïssé Games and First-Order Logic
   5.3. Pebble Games and Fixed-Point Logics
   5.4. 0-1 Laws in Finite Model Theory
6. Logic and Databases
   6.1. Database Query Languages
   6.2. Constraints in Databases

**Modal Logic and Its Applications**

Kit Fine, *Professor of Philosophy and Mathematics, New York University, USA*

1. Introduction
2. Language and Logic
   2.1. Language
   2.2. Formulas
   2.3. Systems
   2.4. Meta-theorems
3. Semantics
   3.1. Models
   3.2. Truth
   3.3. Validity
4. Soundness and Completeness for K
5. Some Other Systems
   5.1. Completeness for T
   5.2. Standard Extensions
   5.3. Completeness of Standard Extensions
6. Some Other Results
   6.1. Strong Completeness
   6.2. Finite Frame Property
   6.3. Decidability
   6.4. Some General Questions
7. Alternative Interpretations of ‘~ ’
   7.1. Alethic Modality
   7.2. The Epistemic Interpretation
   7.3. The Tense-logical Interpretation
   7.4. The Deontic Interpretation
8. Multimodal Logics
   8.1. Tense and Epistemic Logic
   8.2. Dynamic Programming Logic
9. Non-standard Semantics
   9.1. Relevance Logic
   9.2. Counterfactual Logic
10. Modal Predicate Logic
   10.1. The Problem of Quantifying In
   10.2. Objectualism
   10.3. Conceptualism
   10.4. Counterpart Theory
   10.5. Variable Domains
   10.6. Completeness
11. Modality and Language
   11.1. Meaning
   11.2. Intensional Constructions
   11.3. Indexicality

A Basic Example of NonLinear Equations: The Navier-Stokes Equations 116
University of Paris 7 and Laboratoire Jacques Louis Lions (Univ. Paris 6), France

1. Scaling, hierarchies and formal derivations
2. Stabilities and instabilities of macroscopic solutions
3. Turbulence, weak convergence and Wigner measures
4. Some special properties of the dimension 2

Calculus of Variations, Partial Differential Equations, and Geometry 143
Fabrice Bethuel, Universite Pierre et Marie Curie, Paris, France

1. Introduction
   1.1. Generalities
   1.2. Parameterization of Geometrical Problems
2. An example: minimal surfaces
   2.1. Graphs
   2.2. Conformal Parameterization
   2.3. Bubbles
3. Phase transitions and interfaces
3.1. Ginzburg-Landau Functionals
3.2. The Scalar Case
   3.2.1. The Case $N = 1$
   3.2.2. The Higher Dimensional Case

Linear Differential Equations 161
Louis Boutet de Monvel, Université Pierre et Marie Curie, France

1. Introduction
2. Linearity and Continuity
   2.1. Continuity
   2.2. Linearity
   2.3. Perturbation Theory and Linearity
   2.4. Axiomatically Linear Equations
      2.4.1. Fields: Maxwell Equations
      2.4.2. Densities on Phase Space in Classical Physics
      2.4.3. Quantum Mechanics and Schrödinger Equation
3. Examples
   3.1. Ordinary Differential Equations
   3.2. The Laplace Equation
   3.3. The Wave Equation
   3.4. The Heat Equation and Schrödinger Equation
   3.5. Equations of Complex Analysis
      3.5.1. The Cauchy-Riemann Equation
      3.5.2. The Hans Lewy Equation
      3.5.3. The Mizohata Equation
4. Methods
   4.1. Well posed Problems
      4.1.1. Initial Value Problem, Cauchy-Kowalewsky Theorem
      4.1.2. Other Boundary Conditions
   4.2. Distributions
      4.2.1. Distributions
      4.2.2. Weak Solutions
      4.2.3. Elementary Solutions
   4.3. Fourier Analysis
      4.3.1. Fourier Transform
      4.3.2. Equations with Constant Coefficients
      4.3.3. Asymptotic Analysis, Microanalysis

Differential Equations and Symplectic Geometry 195
J. J. Duistermaat, Department of Mathematics, Utrecht University, The Netherlands

1. Lagrangian Mechanics
2. Hamiltonian Systems and Symplectic Geometry
3. Nonlinear First order Partial Differential Equations
4. Oscillatory Integrals
5. Fourier Integral Operators

From the Atomic Hypothesis To Microlocal Analysis 213
Claude Bardos, University Denis Diderot, France
Louis Boutet de Monvel, Université Pierre et Marie Curie, France

1. Introduction
2. The Schrödinger Equation And Semiclassical Analysis
   2.1. Schrödinger equation
2.2. WKB Asymptotics
2.3. Newtonian and Hamiltonian Mechanics
2.4. Caustic
2.5. Feynman Integral
2.6. Stationary Phase
2.7. ħ Fourier Integral Operator

3. High Frequency Asymptotics and Microlocal Analysis
3.1. Differential Operators
3.2. Microsupport
3.3. Pseudo-differential Operators
3.4. Symbolic calculus, principal Symbol
3.5. Symplectic geometry
3.6. Fourier Integral Distributions, Fourier Integral Operators
3.7. Models, propagation of singularities
3.8. Eigenvalues of elliptic operators
3.9. Miscellaneous
   3.9.1. Microlocal regularity
   3.9.2. Weyl Calculus
   3.9.3. Analytic Pseudo-differential Operators and Analytic Wave Front set
   3.9.4. Gevrey Classes
   3.9.5. Uncertainty principle
   3.9.6. Carleman estimates

Discrete Mathematics 250
Kazuo Murota, University of Tokyo, Tokyo, Japan

1. Introduction
   1.1. Fundamental Issues
   1.2. Squares
   1.3. Graphs
   1.4. Algorithms
2. Bipartite Matchings
   2.1. Matchings and Covers
   2.2. Dulmage-Mendelsohn Decomposition
3. Discrete Convex Functions
   3.1. Definitions
   3.2. Convexity
   3.3. Matroids and Submodular Functions
   3.4. Graphs and Networks
   3.5. Polyhedra and Conjugacy
   3.6. Optimization and Duality
   3.7. Algorithms and Computational Complexity

Graph Theory 294
Hikoe Enomoto, Hiroshima University, Higashi-Hiroshima, Japan

1. Introduction
2. Degrees and Distances
3. Connectivity
4. Operations
5. Trees
6. Factor Theory
7. Eulerian Circuits and Hamiltonian Cycles
8. Coloring
9. Planar Graph
Combinatorics 314
Takeshi Tokuyama, Tohoku University, Sendai, Japan

1. Introduction
   1.1. Set Systems and Bit Sequences
   1.2. Level Set and Combination Numbers
   1.3. Inclusion-Exclusion Principle
   1.4. Recursive Formulas and Asymptotic Bounds
   1.5. Generating Functions and Binomial Formula

2. Selected Topics in Combinatorics
   2.1. Hypergraphs
      2.1.1. Graphs and Hypergraphs
      2.1.2. Transversal, Matching, and Covering
      2.1.3. Pigeonhole Principle and Ramsey Theory
      2.1.4. Block Design
      2.1.5. Codes and Hypergraphs
   2.2. Partially Ordered Set
      2.2.1. Chain and Antichain of a Poset
      2.2.2. Möbius Inversion Formula
   2.3. Matroids
      2.3.1. Axiom and Examples of Matroids
      2.3.2. Matroids and Combinatorial Optimization
   2.4. Combinatorial Geometry
      2.4.1. Arrangements and Convex Polytopes
      2.4.2. Semispaces and K-Sets
      2.4.3. Helly Type Theorems
   2.5. Theory of Partitions
      2.5.1. Partitions and Young Diagrams
      2.5.2. Number of Partitions and Partition Functions
      2.5.3. Young Tableaux

Computational Complexity 339
Osamu Watanabe, Tokyo Institute of Technology, Tokyo, Japan

1. Introduction
2. Machine Models and Complexity Measures
3. Complexity Classes
4. Fundamental Results and Questions
   4.1. Inclusion by Simulation
   4.2. Important Open Questions
   4.3. Comparison of Hardness by Reducibility
5. Selected Topics

Optimization 366
Toshihide Ibaraki, Kyoto University, Kyoto, Japan

1. Introduction
2. Integer Programming
   2.1. Definitions
   2.2. Total Unimodularity
   2.3. Integer Polyhedron
3. Enumerative Algorithms for Integer Programming
   3.1. Branch-and-Bound Algorithms
   3.2. Branch-and-Cut Algorithms
4. Solvable Cases of Integer Programming
5. Approximation Algorithms
5.1. Approximation Algorithms with Performance Guarantees
5.2. Polynomial Time Approximation Schemes
6. Metaheuristics
   6.1. Local Search
   6.2. Fundamentals of Metaheuristic Algorithms

Index 395

About EOLSS 401