THE GRAVITATIONAL TWO-BODY PROBLEM

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Keywords: Two-body problem, Keplerian solution, distributed bodies, rotational motion, translational motion.

Contents

Summary

The gravitational two-body problem is defined and described. The classical Keplerian solution for the motion of two point masses is just one specialized version of this problem, and in general the only one which is completely integrable. This chapter will provide a general definition of the two-body problem making no assumptions on the form of the mass distributions that are mutually gravitating. Then various levels of approximation will be introduced, describing the constraints and general results which exist for this problem. The general, or full, two-body problem actually couples rotational and translational motion in the general case, forming a non-integrable problem. Despite this, there are strong constraints on the Hill and impact stability of this problem. In addition, relative equilibria and their stability can be discussed in a general setting. The chapter culminates in a derivation of the Keplerian solution for the dynamics of two point mass bodies orbiting each other.

1. Introduction

The most basic problem in celestial mechanics is the gravitational two-body problem, specifically the motion of two mutually attracting mass distributions. The understanding of the simplest version of this problem occurred with Copernicus' placement of the sun at the center of the solar system and Kepler's subsequent first description of the elliptical motion of the planets in the solar system. What Kepler discovered based on observation and deduction was a special solution to the general gravitational two-body problem, where the bodies in motion can be approximated as spheres or point masses.

Kepler's solutions provided the descriptive geometry of planetary motion. These results were not placed onto a firm mechanical basis until Newton's celebrated development of the law of gravitation, his laws of motion, and the calculus. Taken together these advancements provided a complete development of Kepler's solutions based on physical principles. Beyond this significant advance, Newton's work also enabled the complete gravitational two-body problem to be fully posed and analyzed. Details of the dynamics and solutions of the unapproximated problem still required significant development and insight, represented by many of the greats of celestial mechanics, physics and mathematics.

This chapter addresses this most fundamental problem in a general setting, placing an emphasis on recent scholarship and on developing a unified view of the two-body problem. Most striking, it is noted that the full two-body problem is not solvable in most of its statements, and can only be solved under some very restrictive approximations consistent with Kepler's original solution. A complete derivation of Kepler's solution is given at the close of the chapter, but first we develop and focus on the more general statement of this problem and detail the constraints, solutions and approximations that enable these problems to be understood. This general approach has been motivated by recent scholarship on the general, or so-called full, two-body problem that accounts for the coupled rotational and translational motion of the two mass distributions. Early work that focused on the coupling between translational and rotational motion traces back to Cassini's Laws on the motion of the moon, although in these cases the librational rotation is driven by the orbit, while the influence of the rotation on the orbit is generally neglected. Starting in the 1950s, Duboshin studied the dynamics of coupled rotational and translational motion and stated the general set of differential equations and their fundamental integrals of motion. In the 1970s Kinoshita developed perturbation theories for these problems under the assumption of relatively weak coupling between the different modes of motion. In the 1990s research into the generalized version of this problem was explored by a number of researchers. Wang, Krishnaprasad, and Maddocks approached the problem from a geometrical mechanics approach and explored the existence and stability of relative equilibria, albeit under the assumption that one of the bodies was a sphere. Maciejewski considered the fully general problem and explored the properties of relative equilibria as well as several different ways to pose the problem. In the 2000s a series of articles by Scheeres explored constraints on the solutions to the full two-body problem and applied this problem to the dynamics of binary asteroids and the evolution of rubble pile asteroids. Significant advances in the study of the averaged full two-body problem was made by Boué and Laskar, generalizing the classical Cassini states and showing how they fit into a larger set of integrable motions for the averaged problem.

The approach taken in this chapter is focused on sharp results that do not make strong assumptions on the motion, such as are found in averaging theories. Solutions which yield general and particular solutions to the problem are given, and constraints for the system which act on the full solution space are emphasized. The goal of the current chapter is to develop a consistent and unified approach to this problem, relying on classical mechanics formulations. Several theorems are introduced and developed to capture key results that hold for the general system and some special subsystems. There are several results for the general full two-body problem that are not well known and are somewhat surprising and at odds with the classical Kepler solution.

First is that the full two-body problem is in general non-integrable and can exhibit chaotic motion (although we will not establish these results here). This occurs either due to the coupling of rotational motion of the two bodies with their relative translational motion or due to the non-spherical mass distributions of either one of the bodies. Second, for the general evolution of the two-sphere problem is that a system with a fixed value of angular momentum can have multiple circular orbits, some of which can be unstable. Only when the system is limited to two point-mass distributions does it become a fully integrable problem. In the following we work through a number of these different results and only arrive at the integrable version of the problem in the last section.

2. Body and Mass Distribution Specifications

The core assumption we make in this chapter is that the mass distributions of the two bodies are rigid, meaning that we do not account for any deformation in their shape or mass distribution. Figure 1 provides a graphical definition of the problem, with the following section providing a mathematical description.

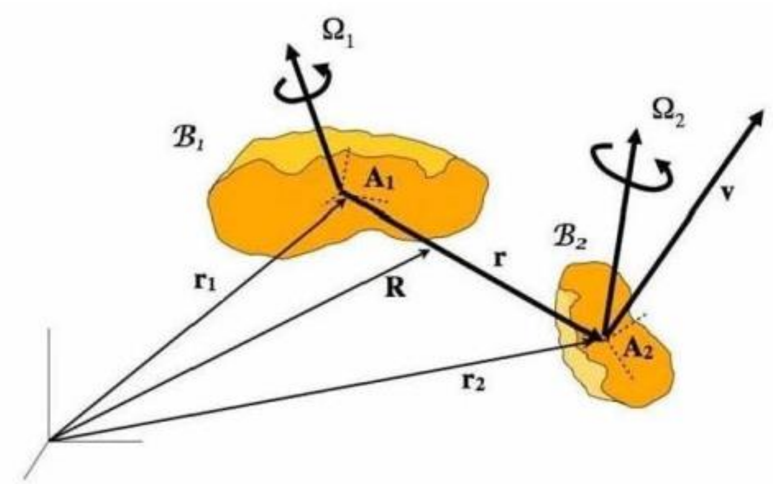

Figure 1. The full gravitational 2-Body Problem and its degrees of freedom

2.1. Mass and Center of Mass

Assume that there exist two rigid bodies with distributed masses, characterized by having finite densities and well defined limits. Their differential masses are defined as

$$dm_i = \sigma_i(\boldsymbol{\rho})dV \tag{1}$$

where σ_i is the varying density of the ith body, $i = 1, 2$, and $\boldsymbol{\rho}$ is the position of the mass element referenced to some frame. The density is zero when $\boldsymbol{\rho}$ is taken outside of the body and is finite within the body, denoted as $\mathcal{B}_i$. With this definition the total mass of each body equals

$$M_i = \int_{\mathcal{B}_i} dm_i \tag{2}$$

and the location of each of the body's center of mass is computed as

$$\mathbf{r}_i = \frac{1}{M_i} \int_{\mathcal{B}_i} \boldsymbol{\rho}\, dm_i \tag{3}$$

The joint mass distribution of the system is defined via the differential mass element

$$dm = dm_1 + dm_2 \tag{4}$$

and the joint bodies as $\mathcal{B}\{\mathcal{B}_1, \mathcal{B}_2\}$. This allows the total mass of the system to be defined as

$$M_1 + M_2 = \int_{\mathcal{B}} dm \tag{5}$$

The barycenter of the system is then found as

$$\mathbf{R} = \frac{1}{M_1 + M_2} \int_{\mathcal{B}} \boldsymbol{\rho}\, dm \tag{6}$$

As will be discussed later, we can take $\mathbf{R} \equiv \mathbf{0}$ in general.

2.2. Relative Orientations

The orientation of the bodies is defined by rotation dyadics that transform a vector in the body-fixed frame into inertial space, denoted as $\mathbf{A}_1$ and $\mathbf{A}_2$. Then the complete specification of each body in an inertial frame is $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{A}_1, \mathbf{A}_2\}$ and the relative position and orientation of the bodies with respect to each other is $\{\mathbf{r}, \mathbf{A}\}$ where the relative position and attitude of body 2 relative to body 1 is defined as

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \tag{7}$$

$$\mathbf{A} = \mathbf{A}_1^{\mathrm{T}} \cdot \mathbf{A}_2 \tag{8}$$

As each of these terms represent 3 degrees of freedom, to specify the relative position and orientation of two rigid bodies requires a total of 6 degrees of freedom. These are called the internal or relative degrees of freedom. To orient these internal degrees of freedom with respect to an inertial frame requires an additional 3 degrees of freedom, represented by the rotation dyadic $\mathbf{A}_1$. Figure 1 shows a graphical representation of this full system. We note that each of the rigid bodies has an angular velocity $\boldsymbol{\Omega}_i$ that defines the instantaneous rate of rotation between the body-fixed frame and an inertial frame.

2.3. Moments of Inertia

An additional mass distribution quantity of interest is the inertia dyadics of each body, defined by

$$\mathbf{I}_i = \int_{\mathcal{B}_i} \left[\rho^2 \mathbf{U} - \boldsymbol{\rho}\boldsymbol{\rho} \right] dm_i \qquad (9)$$

where $\mathbf{U}$ is the identity dyadic and the product of two vectors, e.g. $\boldsymbol{\rho}\boldsymbol{\rho}$, is a dyad. The coefficients of this dyadic can be computed in closed form for special cases such as constant density spheres, ellipsoids, and polyhedra.

The inertia dyadic of the entire system can also be specified as

$$\mathbf{I} = \int_{\mathcal{B}} \left[\rho^2 \mathbf{U} - \boldsymbol{\rho}\boldsymbol{\rho} \right] dm \qquad (10)$$

and simplifies into the individual inertia dyadics and the inertia dyadic of the two masses if the system is computed relative to the barycenter

$$\mathbf{I} = \frac{M_1 M_2}{M_1 + M_2} r^2 \left[\mathbf{U} - \hat{\mathbf{r}}\hat{\mathbf{r}} \right] + \mathbf{I}_1 + \mathbf{I}_2 \qquad (11)$$

where the hat designates a vector as a unit vector.

Of interest later is the moment of inertia relative to a fixed direction, $\hat{\mathbf{H}}$, computed as

$$I_H = \hat{\mathbf{H}} \cdot \mathbf{I} \cdot \hat{\mathbf{H}} \qquad (12)$$

Note that I_H is a function of the relative position of the two masses and their orientation, all relative to the unit vector $\hat{\mathbf{H}}$.

Also of interest is the polar moment of inertia, computed as one-half of the trace of the total inertia

$$I_{\mathrm{P}} = \frac{1}{2} \mathrm{trace}\left(\mathbf{I} \right) \qquad (13)$$

$$= \frac{M_1 M_2}{M_1 + M_2} r^2 + \frac{1}{2} \mathrm{trace}\left(\mathbf{I}_1 + \mathbf{I}_2 \right) \qquad (14)$$

Note that the polar moment of inertia for a 2-body system is only a function of the separation between the bodies, or $I_{\mathrm{P}}\left(r\right)$, and is independent of the orientation of the two mass distributions. An important inequality can be proven between the polar moment of inertia and the moment of inertia relative to a fixed direction

$$I_H \leq I_\mathrm{P} \qquad (15)$$

which holds for any fixed direction $\hat{\mathbf{H}}$.

2.4. Body Shapes and Geometry

In the following we will assume that both bodies are convex. This is not an essential assumption but makes it simpler to discuss situations when the two bodies can come into contact. If they are both convex, then at every relative configuration of the system there is a well defined minimum distance between the bodies $d(\hat{\mathbf{r}},\mathbf{A})$ defined as the radius at which the two bodies touch for their relative configuration. This distance changes smoothly with $\hat{\mathbf{r}}$ and $\mathbf{A}$ and is constant for all rotations of $\mathbf{A}$ about the unit vector $\hat{\mathbf{r}}$. If non-convex bodies are assumed then there is the potential for multiple distances between the distributions at a given relative configuration and discontinuities in the minimum distance as a function of $\hat{\mathbf{r}}$ and $\mathbf{A}$.

The maximum of these minimum distances can be specified as $D = \max_{\hat{\mathbf{r}},\mathbf{A}} d(\hat{\mathbf{r}},\mathbf{A})$. This quantity is of fundamental interest for any two body system as beyond this distance the two bodies can never impact with each other. This limit can be defined independent of whether the bodies are convex or not.

3. Newtonian Gravitational Attraction

Having defined the two bodies and their mass distribution, we next consider the relative forces that these distributions apply to each other due to Newton's law of gravitational attraction.

3.1. Relative Forces

Newton's fundamental law of gravitational attraction states that two differential mass elements will experience an attraction between them proportional to the product of the masses, inversely proportional to the square of the distance between them and directed along their relative position. Given our definition of the relative position vector $\mathbf{r}$ as going from body 1 to body 2 the differential force that a mass element in body 1 places on a mass element in body 2 is

$$d\mathbf{F}_{12} = -\frac{\mathcal{G} dm_1 dm_2}{\left|\mathbf{r} + \mathbf{A}_2 \cdot \boldsymbol{\rho}_2 - \mathbf{A}_1 \cdot \boldsymbol{\rho}_1\right|^3}\left(\mathbf{r} + \mathbf{A}_2 \cdot \boldsymbol{\rho}_2 - \mathbf{A}_1 \cdot \boldsymbol{\rho}_1\right) \qquad (16)$$

where we recall that $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and $\boldsymbol{\rho}_i$ is the position of a mass element of body i relative to its center of mass. The differential force of a mass element in body 2 on a mass element in body 1 is simply $-d\mathbf{F}_{12}$ in accordance with Newton's law of action and reaction. To compute the total force that body 1 exerts on body 2, and vice-versa, these differentials are integrated over both mass distributions:

$$\mathbf{F}_{12} = \int_{\mathcal{B}_1}\int_{\mathcal{B}_2} d\mathbf{F}_{12} \tag{17}$$

and $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

The differential force can also be derived from a scalar potential function, defined as the differential potential gravitational energy between the bodies

$$dU = -\frac{\mathcal{G}dm_1 dm_2}{\left|\mathbf{r} + \mathbf{A}_2 \cdot \boldsymbol{\rho}_2 - \mathbf{A}_1 \cdot \boldsymbol{\rho}_1\right|} \tag{18}$$

then

$$d\mathbf{F}_{12} = -\frac{\partial dU}{\partial\left(\mathbf{r}_2 + \boldsymbol{\rho}_2\right)} \tag{19}$$

$$d\mathbf{F}_{21} = -\frac{\partial dU}{\partial\left(\mathbf{r}_1 + \boldsymbol{\rho}_1\right)} \tag{20}$$

Note that $\dfrac{\partial dU}{\partial\left(\mathbf{r}_i + \boldsymbol{\rho}_i\right)} = \dfrac{\partial dU}{\partial \mathbf{r}_i}$ in general.

The gravitational potential energy between the bodies is then defined by integrating this differential potential over both bodies

$$U\left(\mathbf{r}_1, \mathbf{r}_2, \mathbf{A}_1, \mathbf{A}_2\right) = \int_{\mathcal{B}_1}\int_{\mathcal{B}_2} dU \tag{21}$$

$$-\mathcal{G}\int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \frac{dm_1\left(\boldsymbol{\rho}_1\right) dm_2\left(\boldsymbol{\rho}_2\right)}{\left|\mathbf{r}_2 - \mathbf{r}_1 + \mathbf{A}_2 \cdot \boldsymbol{\rho}_2 - \mathbf{A}_1 \cdot \boldsymbol{\rho}_1\right|} \tag{22}$$

where the integration variables $\boldsymbol{\rho}_i$ are expressed in their respective body-fixed frames and all terms in the vector magnitude are specified in inertial space. The forces of these bodies on each other are then computed as

$$\mathbf{F}_{12} = -\frac{\partial U}{\partial \mathbf{r}_2} \tag{23}$$

$$\mathbf{F}_{21} = -\frac{\partial U}{\partial \mathbf{r}_1} \tag{24}$$

3.2. Relative Moments

For bodies with finite mass distributions it is also necessary to compute the mutual moments of the force (or moments) that are exerted on each other. The moment that body 1 exerts on body 2 is found by integrating the differential moment over both bodies:

$$d\mathbf{M}_{12} = \mathbf{A}_2 \cdot \boldsymbol{\rho}_2 \times d\mathbf{F}_{12} \tag{25}$$

$$\mathbf{M}_{12} = \int_{\mathcal{B}_1} \int_{\mathcal{B}_2} \mathbf{A}_2 \cdot \boldsymbol{\rho}_2 \times d\mathbf{F}_{12} \tag{26}$$

The moment can also be related to the mutual potential, although the details are more involved. This is found by taking the partial of the mutual potential with respect to the infinitesimal rotations about each axis of body 2, which we represent as $\mathbf{M}_{12} = -U_{\boldsymbol{\theta}_2}$. In practice, once the relative attitude of body 2 is defined explicitly using $\mathbf{A}_2$ the partials with respect to the angles can be computed from this relationship. The moment acting on body 1, $\mathbf{M}_{21}$, is similarly computed as $-U_{\boldsymbol{\theta}_1}$. It must be noted that the mutual moments are not equal and opposite, but that their sum equals the negative moment of the total gravitational force between the bodies

$$\mathbf{M}_{12} + \mathbf{M}_{21} + (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}_{12} = \mathbf{0} \tag{27}$$

meaning that we do not need to solve for one of the moments if the force is known.

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Biographical Sketch

Daniel Jay Scheeres (born in 1963 in Royal Oak, Michigan, USA) has a Bachelor's of Science in Letters and Engineering from Calvin College (1985), a Bachelor's and Master's of Science in Aerospace Engineering from The University of Michigan (1987 and 1988, respectively), and a PhD. in Aerospace Engineering from The University of Michigan (1992) where he studied with Nguyen Xuan Vinh. Since 2008 he has been the A. Richard Seebass Endowed Chair Professor in the Department of Aerospace Engineering Sciences at The University of Colorado Boulder. Prior to this he held academic positions at the University of Michigan and Iowa State University. Prior to that he was a Senior Member of the Technical Staff at the Jet Propulsion Laboratory / California Institute of Technology. He is past chair of the American Astronomical Society's Division on Dynamical Astronomy and the vice-president of the Celestial Mechanics Institute. His research interests include celestial mechanics of distributed bodies, the mechanics of comets and asteroids, the navigation and dynamics of spacecraft, and the long-term evolution of orbit debris.