GRAPH THEORY

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Contents

Summary

This section presents fundamental results in graph theory. After the introduction of basic terminology and concepts such as isomorphism of graphs and adjacency of vertices, a number of identities and inequalities valid for the degrees of vertices in a graph are explained.

This is followed by succinct descriptions of major results in graph theory. The topics treated here include connectivity, factors, Eulerian circuits, and Hamiltonian cycles. Trees and planar graphs constitute two important subclasses of graphs, for which many interesting results such as Nash-Williams' formula and the four color theorem can be obtained.

1. Introduction

Graphs are used as models of various structures (relations, communication networks, electronic circuits, molecular structures etc.).

An (undirected) *graph G* consists of a nonempty set $V(G)$ of elements called *vertices* (or *points* or *nodes*) and a set $E(G)$ of elements called *edges* (or *lines* or *branches*) together with a relation of incidence that associates with each edge two vertices, called its *ends*.

The number of vertices $|V(G)|$ is the *order* of *G*, and the number of edges $|E(G)|$ is the *size* of *G*. A graph is finite if $V(G)$ and $E(G)$ are finite sets. In the following, we deal with finite graphs, so 'graph' means 'finite graph'. A graph can be described by a diagram in which each vertex is represented by a point and each edge is represented by a linear segment or curve joining the points corresponding to its ends.

An edge with identical ends is a *loop*, and two or more edges with the same pairs of ends are *multiple edges* (or *parallel edges*). A graph is called *simple* if it contains neither loops nor multiple edges. If loops and multiple edges are allowed, such a graph is sometimes called a *multigraph*.

Two graphs G and H are *isomorphic* (denoted by $G \cong H$) if there are bijections $f: V(G) \to V(H)$ and $g: E(G) \to E(H)$ such that vertex v and edge e are incident in G if and only if $f(v)$ and $g(e)$ are incident in H. In other words, isomorphic graphs have the same structure. Isomorphic graphs are often identified with each other, and in that case, $G = H$ is used to mean $G \cong H$.

For simple graphs G and H, they are isomorphic if there is a bijection $f: V(G) \to V(H)$ such that vertices u and v and adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. Such f is called an isomorphism from G to H. For example, in Figure 1, if we define $f(x_i) = y_i (1 \le i \le 6)$, f is an isomorphism from G_1 to G_2.

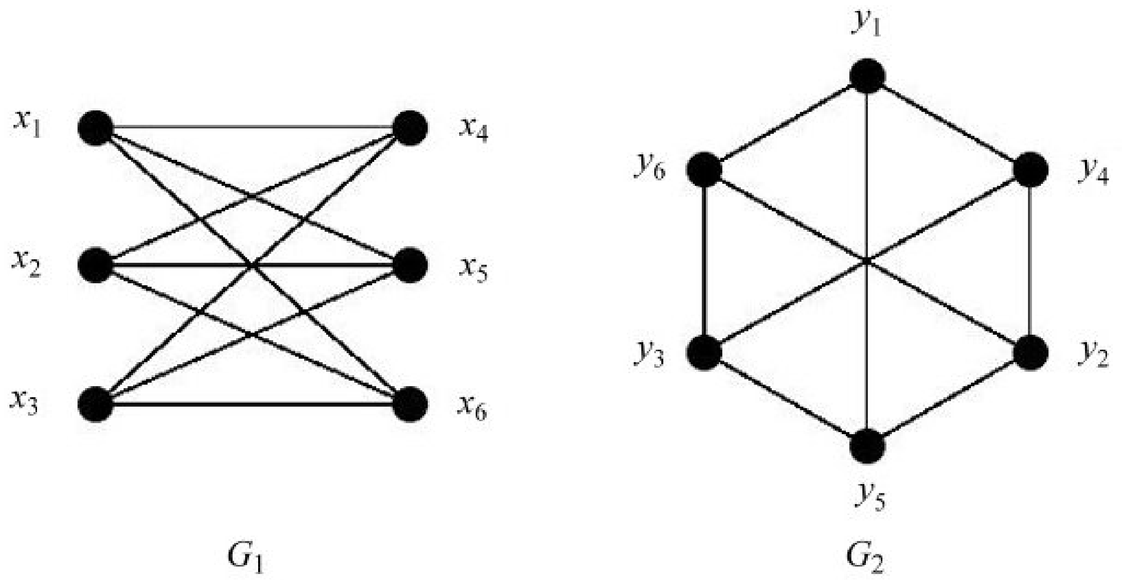

Figure 1. Isomorphic graphs

For a set S, the family of all the subsets of S with cardinality k is denoted by $\binom{S}{k}$. The edge set of a simple graph G can be identified with a subset of $\binom{V(G)}{2}$, and an edge with ends u and v can be identified with $\{u, v\}$, and is often denoted more simply by uv.

The two ends of an edge are said to be *joined* by the edge and to be *adjacent* to one another. Two edges with a common end are also called adjacent. A subset S of $V(G)$

(resp. a subset *F* of $E(G)$) is *independent* if any two vertices in *S* (resp. any two edges in *F*) are nonadjacent.

For two graphs *G* and *H*, *H* is a *subgraph* of *G* if $V(H)$ is a subset of $V(G)$, $E(H)$ is a subset of $E(G)$ and the incidence relation in *H* is the restriction of that in *G*. Also, *G* is called a *supergraph* of *H*. If $V(H)=V(G)$, *H* is called a *spanning subgraph*. If $V(H)=S$ and all the edges of *G* joining vertices in *S* are in $E(H)$, *H* is the subgraph *induced* by *S*, and is denoted by $G[S]$ (or $\langle S\rangle_G$). For a subset *S* of $V(G)$, $G-S$ is the subgraph induced by $V(G)-S$, and for a vertex *x*, $G-x=G-\{x\}$. For a subset *F* of $E(G)$, $G-F$ denotes the spanning subgraph with the edge set $E(G)-F$, and for an edge *e*, $G-e=G-\{e\}$. A graph *G* is called an *empty graph* if $E(G)=\emptyset$, and is called *trivial* if $|V(G)|=1$ and $E(G)=\emptyset$.

If *G* is a simple graph and every two vertices of *G* are joined by an edge, *G* is called *complete*. A complete graph of order *n* is denoted by K_n.

If $V(G)$ is the disjoint union of two subsets *A* and *B*, and every edge joins a vertex in *A* and a vertex in *B*, *G* is called a *bipartite graph*, and *A* and *B* are *partite sets* of *G*.

If *G* is a bipartite simple graph and every two vertices in different partite sets are adjacent, *G* is called *complete bipartite*. A complete bipartite graph with partite sets of order *m* and *n* is denoted by $K_{m,n}$.

Digraphs (directed graphs) are defined similarly: A *digraph D* consists of a nonempty set $V(D)$ (the vertex set) and $E(D)$ (the edge set). Each element in $E(D)$ is called as a (directed) edge or an *arc*, and connects two vertices with a direction.

When there are no multiple edges (that is, two or more edges connecting the same two vertices with the same direction), the digraph is called simple. The edge set of a simple digraph *D* can be identified with a subset of $V(D)\times V(D)$. A simple digraph may contain an edge from *u* to *v* and an edge from *v* to *u*.

When a digraph *D* contains no loop and no such pair of edges (that is, if there is an edge from *u* to *v*, there is no edge from *v* to *u*), *D* is called *oriented*. In other words, an oriented graph can be obtained from a simple undirected graph by assigning a direction to each edge.

The *underlying graph* of a diagraph *D* is the graph obtained from *D* by replacing each directed edge (u,v) or (v,u) by the edge *uv*.

A digraph *D* is called *complete* if for every two distinct vertices *u* and *v* of *D*, at least one of (u,v) or (v,u) is in $E(D)$. The *complete symmetric digraph* of order *n* is a simple directed graph that contains both directed edges (u,v) and (v,u) for any distinct vertices *u* and *v*, and is denoted by K_n^*. A complete oriented graph is called a *tournament*.

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Biographical Sketch

Hikoe Enomoto was born in 1945, studied group theory in University of Tokyo, and got Ph.D. Degree in 1975 by constructing the character tables of Lie groups of type G_2. Now, interested in graph theory such as connectivity, factor theory and factorizations, coloring and labeling. Also interested in computer science and several other combinatorial structures such as difference sets and Hadamard matrices. Gave lectures at Kyoto University for a year, at University of Tokyo for 18 years, at Keio University for 14 years, and moved to Hiroshima University in 2003. Apart from mathematics, interested in contract bridge and marathon.