MATHEMATICAL MODELS OF AGRICULTURAL SUPPLY

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Summary

Environmental protection principles which have emerged in the modern agriculture specify strict limits for the adopted farming systems. At the same time the solution of food supply problem for the growing world population remains an acute task. Therefore, various resource-saving programs acquire a great significance for the sustainable agriculture development. This paper describes an approach for solution of the above mentioned problems, based on mathematical modeling and computer technology approaches.

The optimization problems are generalized for land, fertilizer, irrigation water, facilities, labor use in plant-growing, the herd structure and feeding rations in livestock. Mathematical methods of resource utilization optimization have been used in practice from the end of the Second World War. First mathematical programming approaches include the method of linear programming (simplex method). From that time, the models of agricultural processes and the optimization methods have been developed quickly. At the moment deterministic statements of problems prevail, but the number of problems increases with allowance for uncertainty elements connected with weather and climate solved by methods of stochastic programming. Agroecosystem management needs rather complex dynamical models. Based on the specific problem or the specific parameters a wide range of methods from classical analytical studies of the systems of

differential equations to numerical simulation was suggested within the expert systems of decision making support. Managing field variability by using information technology tools for handling information offer a possibility to solve the optimization task based on the development of site-specific farming methods in agriculture.

1. Introduction

The utilization of natural resources in the existing technologies of the agriculture management is close to the saturation condition. The exponential growth of irreplaceable energy, water and fertilizer consumption is observed per every extra production unit. Simultaneously, the supply of essential natural resources, necessary for food production, steadily decreases per capita.

Therefore, various resource-saving programs (lands, fresh water, energy raw materials, phosphoric and potassium fertilizers, etc.) acquire a great significance for sustainable agriculture development. The methods of enhancement of resource utilization efficiency in agriculture based on application of mathematical models of the corresponding processes and optimization methods play the key role in these programs.

Modern computer expert systems for supporting decision making in agriculture based on developed and complex dynamical models of agroecosystem productivity are the key systems in this field of studies. There is a natural transition from linear to nonlinear objective functions and restrictions and from statistical to dynamical models. The role of different stochastic problems of resource allocation optimization connected with the application of current and prognostic information, in particular meteorological, steadily increases.

One more important point is practical realization of ideas of the so-called precision (exact) farming which results in a sharp rise in the volume of applied information (increase in the space and time resolution of data on soil conditions). These data are used to enhance the efficiency of fertilization, to optimize the plant protection means.

The decision of optimization problems is facilitated by available software packages realizing different sets of numerical algorithms of optimization. The availability of these packages, however, is a necessary but not sufficient condition for an effective solution of new problems. The analysis of physical peculiarities of applied models and the possibility of utilization of the methods of mathematical analysis, including classical ones, are also essential. It is very important for more thorough understanding of the essence of problems to be solved where the use of standard packages promotes only a little.

2. Models and decision making in agriculture

Problems of resources allocation arise in all cases when it is essential to implement a variety of activities under the restrictions associated either with resources availability or with the possibility of their utilization. In these situations the problem of optimal resources allocation between the forms of activities is important. Many such problems can be solved by mathematical programming or other related methods. It is assumed

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here that all parameters in the economic-mathematical model (resources, technical-economic coefficients and coefficients of the objective function) are deterministic, *a priori* known quantities. Problems considering the risk and uncertainty factors, in which some (and in the general case all) parameters are random values, are known as stochastic problems of resources allocation. Stochastic is the integral attribute of agricultural production connected with the uncertain character of the environment and, primarily, weather conditions.

Consider the statement of a general problem of linear programming in a matrix form: find

$$F(x) = C^{*}X \to \max \text{ at } AX < B,\ X > 0.$$

Here the matrix *A* and the vectors *B* and *C* are deterministic; it is the most frequently encountered statement of a problem. In stochastic problems A, *B* and *C* may be random. Problems of stochastic programming differ considerably in the objective function. The following objective functions are used:

a) Mathematical expectation of the quantity in a linear form:

$$F(x) = M\left[C^{*}X\right] \to \max(\min),$$

where *M* is the symbol of mathematical expectation. It is the primary objective function for problems, the optimization criterion of which is the maximum gross production, maximum profit, etc.

b) The probability that the linear form exceeds a certain fixed level:

$$P[C^{*}X > k]\ k] \to \max,$$

where *k* is a given number. For example, the problem is to find the maximum probability that the profit will be not less than a given number *k*.

c) Dispersion of a linear form:

$$X^{*}DX \to \min,$$

where *D* is the quadratic matrix, whose elements are the yield dispersions and covariances, X^{*} is the transposed vector *X*. This optimization criterion is reduced to the option of a plan, in which dispersion of a certain index (gross production, income) has minimum value. Decrease in the effective index variability of agricultural production, i.e. its stability rise seems to be extremely important.

d) Linear combination of mathematical expectation and dispersion of linear form:

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$$C^{*}X - \gamma X^{*}DX \rightarrow \max,$$

where γ is the penalty for dispersion unit. Here mathematical expectation of a linear form is maximized, however, the penalty for dispersion unit is introduced. When using this criterion of optimality, the choice of γ , i.e. the objective assessment of dispersion effect on the plan quality is difficult.

For solving problems in stochastic programming two approaches are possible: the desired plan of the problem can be considered as a deterministic or random vector. In the first case it means that the decision obtained is applied constantly (e.g. during some years despite observed agrometeorological conditions). The problem may be formulated in such a way that the decision obtained earlier is corrected or reconsidered depending on the observed conditions and/or results of forecasting. The specific importance of such problems keeps growing, with the prospects of application of flexible technologies, adapted and changed in conformity to forecaster conditions.

The next example illustrates the case when the decision of the problem is beyond the limits of mathematical programming. Table 1 presents the income matrix characterizing cultivation efficiency of four crops k differed in their water requirements.

F	K			
	K_1	K_2	K_3	K_4
F_1	5	0	-1	-2
F_2	2	3	1	-1
F_3	-4	1	5	-1
F_4	-1	0	1	1

Table 1. Cultivation efficiency (conditional unit) of different crops depending on water supply

Positive numbers here correspond to the cases when crop cultivation yields a good return, and negative numbers show losses.

F	Π				Σ
	Π_1	Π_2	Π_3	Π_4	
F_1	0.05	0.02	0.02	0.01	0.10
F_2	0.03	0.10	0.03	0.04	0.20
F_3	0.02	0.03	0.03	0.05	0.40
F_4	0.05	0.03	0.02	0.20	0.30
Σ	0.15	0.18	0.37	0.30	1.0

Table 2. Conjugation matrix for four-phase forecast of summer precipitation

Imagine that a farmer obtains the forecast of summer precipitation done in good time and, depending on this forecast, decides what crop is preferable. Forecast characteristics and climatic recurrences of different gradations of summer precipitation are presented in Table 2.

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Multiplication of the income matrix

$$\left|U_{ij}\right|_{4x4} = \begin{vmatrix} 5 & 2 & -4 & -1 \\ 0 & 3 & 1 & 0 \\ -1 & 1 & 5 & 1 \\ -2 & -1 & -1 & 1 \end{vmatrix}$$

by the conjugation matrix presented in Table 2

$$\left|P_{ij}\right|_{4x4} = \begin{vmatrix} 0.05 & 0.02 & 0.02 & 0.01 \\ 0.03 & 0.10 & 0.03 & 0.04 \\ 0.02 & 0.03 & 0.03 & 0.05 \\ 0.05 & 0.03 & 0.02 & 0.02 \end{vmatrix}$$

gives the matrix-product

$$\left|U_{ij}\right|_{4x4} = \begin{vmatrix} \underline{0.18} & 0.15 & -1.06 & -0.27 \\ 0.11 & \underline{0.33} & 0.39 & 0.17 \\ 0.13 & 0.26 & \underline{1.53} & \underline{0.48} \\ -0.10 & -0.14 & -0.35 & 0.09 \end{vmatrix}.$$

The maximum elements in each column are underlined here. Considering the position of these elements in lines, the following rule can be formulated for optimal decision making in the economy:

- with small predicted quantity of precipitation (gradation F_1) it is desirable to cultivate the most drought-resistant crop K_1 ;
- if moderate precipitation is expected over the year (F_2) the crop K_2 will be the most rewarding;
- in the event of forecasts of heavy (F_3) and very heavy (F_4) precipitation the crop K_3 is to be cultivated.

So, in three of the four cases it is profitable to trust the forecast; in the fourth it is desirable to cultivate the crop K_3 instead of the crop K_4 according to the forecasts.

By summing the maximum elements in a matrix-product, calculation of the mean income under an optimal strategy application gives 2.52 conditional units for a farm whereas with full reliance on the forecast this value would be 2.13 conditional units.

In the case when the mathematical model is dynamic and can be presented by differential equations, the corresponding problems of management can be solved by classical methods, e.g. using the Pontryagin maximum principle. The dynamics of grass biomass under multi-cutting management for hay making was described by Sirotenko:

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$$\frac{dm}{dt} = D\frac{m - m^2}{m + F}, \quad m(0) = m_0,$$

$$D = \frac{Ro}{1 + Rr}, \; F = \frac{Ro}{k \cdot \alpha \cdot b - Ro}, \; m = \frac{M}{M_{\max}},$$

where M is the dry crop biomass and M_{max} is its maximum value, D, R_0, Rr, α, k and b are constants, $m = m_0$ when $t=0$.

The following task of optimal management is set:

$$\int_0^T m(t)u(t)dt \to \max,$$

$$\frac{dm}{dt} = f(m) - U \cdot m, m_0 = m(T) = m(0), 0 \le u < U .$$

Here:

$$f(m) = D\frac{m - m^2}{m + F}, \; f(0) = f(1) = 0, \; \frac{df}{dm} = 0 \text{ at } m = \mu = \sqrt{F(1+F)} - F .$$

Management *u(t)* is the specific rate of " haying"of phytomass, and *U* denotes the maximum rate.

Shlyachkova showed that decision of the task is possible based on the Pontryagin maximum principle:

$$U(t) = \begin{cases} 0, & t \in (0, \tau_{\max}), \\ f(\mu)/\mu, & t \in (\tau_{\max}, t_T), \\ U_m, & t \in (t_T, T), \end{cases}$$

where $\tau_{\max}$ is the time during which the rate of crop growth reaches maximum, t_T denotes the beginning of the time period appropriate for the maximum number of cuttings , T is the end of the cutting period.

Another interesting example of management is proposed by Noy Meir I, who formulates the following equation for pasture biomass dynamics

$$\frac{dV}{dt} = gV(1 - \frac{V}{V_m}) - C_m \frac{V - V_r}{(V - V_r) + (\;\; V_k - V_r)} H,$$

where V, V_m are, respectively, the current green biomass and its maximum value, kg m$^-$

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^{2} ; g is the maximum specific growth V, day^{-1} ; C_m is the maximum rate of biomass consumption, kg (animal per day)$^{-1}$;V_r is the biomass of unpalatable crop residuals, kg m^{-2} ; V_k is the Michaelis-Menthen constant specifying biomass equal to ½ of that required for an animal intake to be sated ; H is the grazing capacity, animals m^{-2}.

The stability of the system under permanent pasture regime is verified from the balance condition $\frac{dV}{dt}=0$ with determination H and V values at which this balance is possible. It turned out that

$$H^* = \frac{gV^*(V_m - V^*) \ \ (V^* + V_k - 2V_r)}{C_m V_m (V^* - V_r)}$$

and the system has the only point of balance

$$H^* < \frac{gV_K}{C_m},$$

which is evaluated as "safe capacity". The latter term denotes permissible carrying capacity of the pasture which can guarantee absence of manifestation of system degradation phenomena.

The methods of decision making in agriculture with available simulation models should be considered in particular. These models can be used together with numerous mathematical methods of economic analysis, e.g. linear and dynamic programming, models of irrigation regime optimization, and models of risk and uncertainty analysis. Consider here the application of EV and MGSD analyses. So, in two plans A and B, for which mathematical expectations are $E(A)$, $E(B)$ and square deviations are $V(A)$, $V(B)$, respectively; A exceeds if

$$E(A) = E(B) \quad \text{and} \quad V(A)<V(B)$$

or if

$$V(A) = V(B) \ \text{and} \ E(A)>E(B).$$

The plan A is said to be more *EV*- effective than B. The attractiveness of a plan increases with E and reduces with V. To compare different plans they can be presented in the EV diagram where E is the ordinate and V is the abscissa. The EV-analysis of risk assumes that the objective function (income, yield, etc.) follows the normal or at least symmetric law of probability distribution.

Of two plans A and B the plan A is more preferable, i.e. the plan A is more effective according to the MGSD analysis, if

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$$E(A) \geq E(B) \quad \text{and} \quad E(A) - \Gamma(A) \geq E(B) - \Gamma(B),$$

where $\Gamma(A), \Gamma(B)$ are the Gini coefficients of distributions A and B.

The Gini coefficient of a property equals to one-half of the mean Gini difference which denotes the mean absolute difference for a couple of values of the variable (the number of differences for the matrix row with *n* elements is equal to $\frac{1}{2} n(n-1)$). This approach based on Gini coefficients has certain advantages as compared to the EV-analysis.

The dynamical CERES-Maize model was used by Rithveet for solving the problem of optimizing the dates of sowing and fertilization for maize in Gainesville, USA. Meteorological information for 10 years was used in 6 versions of model simulation runs (for two sowing dates – 8 March and 8 April and three rates of fertilizers), 6 x 10 = 60 runs in all. Numerical experiments have shown that the maximum yield and the maximum income are expected at the early date of sowing and fertilization of 60 kg ha^{-1}. This strategy, however, is more risky as compared to that of late sowing with similar fertilization of 60 kg ha^{-1} (if for the first version $E(x) - \Gamma(x)$ = 195, for the second one this index is 372). Almost zero probability of losses (negative profit) also shows the advantages of the second version whereas for the first version this possibility is estimated as 8% (table 3).

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Biographical Sketches

Oleg D.Sirotenko is Head of Department of the All Russian Institute of Agricultural Meteorology. His main scientific interests include mathematical modeling of energy-mass exchange in the soil-plant-atmosphere system and development of models describing weather and climate influence on crop productivity, and assessment of global climatic change and greenhoouse impact on agriculture. He is the lead author on the assessment by the IPCC (Intergovermental Panel on Climate Change) of potential impact on agriculture, the author of the IIASA/UNEP research project on climate change and agriculture. He is Chairman of the Working Group on Relationship between Climate and Sustainable Agricultural Production within the WMO Commission for Agricultural Meteorology.

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