MATHEMATICAL MODELS

Jerzy A. Filar
Centre for Industrial and Applicable Mathematics, University of South Australia, Mawson Lakes, Australia

Keywords: Mathematical modeling, mathematical models, uses of mathematical modeling, life support systems and mathematical modeling

Contents

1. Introduction
2. Why Do We Resort to Mathematical Modeling of Life Support Systems?
3. What Kinds of Life Support Systems Can Be Described by Mathematical Models?
4. How Is Mathematical Modeling Done?
4.1. Modeling on the Basis of Previously Established "Governing Equations"
4.2. Extracting Models from Data
4.3. Mathematical Computer Models
4.3.1. Models Created with the Help of High-Level Languages
4.3.2. Integrated Mathematical Computer Models
4.3.3. Complex Adaptive Systems
4.4. Hybrid Mathematical Models
4.5. The Iterative Nature of Model Construction
5. Understanding Uncertainty Accompanying Mathematical Models
6. The Impact of the Information Technology "Revolution" on Both the Practice and Uses of Mathematical Modeling
6.1. Misinformed Use of Mathematical Modeling tools
6.2. A Threat to the "Principle of Parsimony"
6.3. Shadows of Computer Implementations
Acknowledgments
Bibliography
Biographical Sketch

Summary

The societal reliance on mathematical models to support planning, technological innovation, engineering design, and business and development practices is greater than ever before in the history of civilisation. Furthermore, as availability of high-speed computing increases, this trend can only continue. Therefore, the question addressed in this contribution is not whether mathematical modeling is valuable or desirable—that is taken as self-evident—but rather, what are the key principles of best practice when mathematical models of life support systems are developed, implemented and used? The latter can, perhaps, be best understood by examining the independent nature of mathematical systems constituting a model and the consequent limitations when outputs of the model are applied in the "real world."

It is in this context that, in this introduction to the "Mathematical models" theme, we discuss three main questions:

1. Why do we resort to mathematical modeling of life support systems?
2. What types of life support systems can be described by mathematical models?
3. How is mathematical modeling done in, at least, the broadest possible conceptual terms?

1. Introduction

We begin this theme with the following excerpt from the famous "Cave Allegory" (from *The Republic* by Plato, 360 B.C., translated by Benjamin Jowett).

AND now, I said, let me show in a figure how far our nature is enlightened or unenlightened: -–Behold! human beings living in a underground den, which has a mouth open towards the light and reaching all along the den; here they have been from their childhood, and have their legs and necks chained so that they cannot move, and can only see before them, being prevented by the chains from turning round their heads. Above and behind them a fire is blazing at a distance, and between the fire and the prisoners there is a raised way; and you will see, if you look, a low wall built along the way, like the screen which marionette players have in front of them, over which they show the puppets.

I see.

And do you see, I said, men passing along the wall carrying all sorts of vessels, and statues and figures of animals made of wood and stone and various materials, which appear over the wall? Some of them are talking, others silent.

You have shown me a strange image, and they are strange prisoners.

Like ourselves, I replied; and they see only their own shadows, or the shadows of one another, which the fire throws on the opposite wall of the cave?

True, he said; how could they see anything but the shadows if they were never allowed to move their heads?

And of the objects which are being carried in like manner they would only see the shadows?

Yes, he said.

And if they were able to converse with one another, would they not suppose that they were naming what was actually before them?

Very true.

And suppose further that the prison had an echo which came from the other side, would they not be sure to fancy when one of the passers-by spoke that the voice which they heard came from the passing shadow?

No question, he replied.

To them, I said, the truth would be literally nothing but the shadows of the images.
That is certain.

The rationale for selecting the above excerpt from *The Republic* is that it describes in a visual and emotive way what is arguably the essence of the challenge facing most of the modern era researchers involved in the mathematical modeling of life support systems. The challenge is that of creating a model whose outputs—Plato's shadows of images—correspond very closely (under a wide spectrum of inputs) to the measurements of the outputs of the real phenomenon being studied. For instance, a sound model of the spread of an epidemic in a population should be able to estimate the sizes of the different cohorts affected by the disease, at various stages of the epidemic. And yet the mathematical modeling cognoscenti will be conscious of the fact that even a best model of an epidemic is essentially distinct from the epidemic itself. It is more like a wooden figure of an animal in Plato's parable than the animal itself.

Despite the preceding cautionary allegory, the purpose of this contribution is to provide an introduction to the tremendous power of mathematical models—when properly applied—to provide insight to and understanding of many important phenomena. Nowadays, the success of mathematical models and their computer implementations is well documented and spans a wide spectrum of applications from image reconstruction in medical tomography, through spread of pollution in porous media, mathematical models for weather forecasting, to traffic flow models or large-scale production planning models.

The societal reliance on mathematical models to support planning, technological innovation, engineering design, and business and development practices is greater than ever before in the history of civilisation. Furthermore, as availability of high-speed computing increases, this trend can only continue. Therefore, the question addressed in this contribution is not whether mathematical modeling is valuable or desirable—that is taken as self-evident—but rather: *What are the key principles of best practice when mathematical models of life support systems are developed, implemented and used*? The latter can, perhaps, be best understood by examining the independent nature of mathematical systems constituting a model and the consequent limitations when outputs of the model are applied in the "real world."

Interestingly, perhaps, the spectrum of applications is far wider than the spectrum of mathematical techniques used to generate these applications. For instance, a system of ordinary differential equations can be used to adequately model a range of very disparate phenomena (for example, a population of a colony of insects, or harmonic motion of a mass bouncing on a spring). Therein lies one of the great efficiencies of mathematical modeling: the understanding of a relatively small number of mathematical techniques enables one, at least in principle, to model and understand a vast array of phenomena. Nonetheless, researchers employing these techniques must be vigilant to never forget that—no matter how well a model fits the observed data—at a most fundamental level it is still a mathematical object whose "allegiance" is to the internal consistency of the mathematical system and not to the external phenomenon that the researcher wishes it to model.

Thanks to the above-mentioned efficiency of mathematical methods, it would have been possible to structure this theme around a strictly mathematical partition of the most widely used techniques such as algebraic equations, differential equations, statistical models, probabilistic models, simulation models and others. Each technique could then be discussed in some detail and illustrated with a number of successful applications.

That approach to the "Mathematical models" theme would have minimized overlaps. It might also have appealed to some mathematicians by resembling a curriculum of an undergraduate applied mathematics major, but—in all likelihood—it would have been of very limited use to the diverse community of practitioners, researchers and students interested in the modeling of various aspects of life support systems. The main drawback with such a mathematical, approach to the theme would have arisen from a failure to communicate the context and purpose underlying the mathematical modeling undertaken in various disciplines. For the theme to be useful to a broad audience, a researcher in, say, ecology needs to be able to find an article written in the language used by ecologists and addressing issues relevant to ecologists. Only then will the mathematical models described in such an article communicate their intended meaning to the intended audience.

In view of this, the nine topics of the "Mathematical models" theme represent broad categories of endeavor, relevant to the mission of the *Encyclopedia*, where there is already a large body of literature that exploits mathematical modeling to study phenomena and issues relevant to these topics. Thus for each of these topics there exists a community of practitioners, scholars and users with broad interest in that topic. It is hoped that members of these communities will find it easy, informative and rewarding to scan the *EOLSS* and the theme for the articles that are most relevant to them.

Inevitably, this "user oriented" approach to the "Mathematical models" theme results in some inefficiencies and duplication. For instance, it would be reasonable to expect mathematical models of forest management to appear both in the topic devoted to biology and ecology and in the topic dealing with food and agricultural sciences. Furthermore, the mathematical methods used to construct these models may appear and be discussed, in necessarily similar terms, in a number of other topics as well. This is accepted as a necessary consequence of the principle that the theme is being developed to serve a wide interdisciplinary audience.

In this introduction to the "Mathematical models" theme, we shall address three main questions:

1. Why do we resort to mathematical modeling of life support systems?
2. What types of life support systems can be described by mathematical models?
3. How is mathematical modeling done in, at least, the broadest possible conceptual terms?

Arguably, the discussion of the above three questions will shed light only on what might be called the "classical" view of mathematical modeling. However, we live in an era where most educated people have easy access to many tools of mathematical modeling embedded in personal computers on their desks. Furthermore, it is an era where interdisciplinary teams regularly develop large mathematical models on a scale that would have been unthinkable until very recently. In some cases, the models themselves generate mathematical expressions that may or may not be detailed in the conventional way of being written down in a published book, manuscript, or even a technical manual. In recent years, the terms "computer models" and "numerical models" have been used frequently to name some of these modern models; thereby suppressing the fact that, at least internally, they consist of (possibly many) mathematical models. The technological advances that made these new classes of models possible open up many exciting opportunities as well as some inherent dangers. While it will be seen that advantages of the technological progress clearly outweigh the disadvantages, it will also be clear that we have entered an era where new issues concerning the nature and practice of mathematical modeling need to be examined and some of the old issues need to be re-examined.

It is in this context that we shall also discuss the very important issues of:

(4) Understanding and managing uncertainty accompanying the use of mathematical models, and
(5) The impact of the information technology "revolution" on both the practice and uses of mathematical modeling.

In subsequent sections items 1–5 listed above will be discussed in more detail.

2. Why Do We Resort to Mathematical Modeling of Life Support Systems?

In the words of R. Isaacs (1979, p. 37), "The human mind is incapable of thinking other than about models." Irrespective of whether one completely agrees with this statement, it would be hard to argue against a proposition that the desire to use models to help reduce the complexity of situations we face is fundamental to our way of thinking and analysis. It appears to be an innate human trait. It is sufficient to observe children playing with toy cars, dolls, or soldiers to be convinced of how natural it is for us to desire simple models of complex things that we encounter.

However, even if we accept as innate the need to develop models so as to simplify complexity, this does not necessarily make the case for the use of mathematical models. After all, there are many other kinds of models that help reduce complexity, such as architects' physical models of new cities, conceptual models such as those often used by psychologists, or evolutionary models used to explain the origins of *Homo sapiens* (for example, single versus multiple origin theories). These are all instances of very useful models of complex life support systems, interpreted broadly, which make little use of mathematics.

The rationale for resorting to mathematical modeling probably stems from the underlying "dual nature" of mathematics as the *science of relations* as well as the *science of quantity*. Thus whenever it is desirable to study both the quantifiable magnitude of effects and their relationship to one another (if any), the use of mathematics is almost an inevitable consequence.

Of course, there is a legitimate argument that the role of mathematics in modeling phenomena is "merely" that of a *language* used to describe the knowledge and understanding of those who observe and study these phenomena. Indeed, this is the case with many mathematical models of physical phenomena. Furthermore, if that were the only role of mathematics in modeling, then it would be possible to argue that as it becomes increasingly easy to encode understanding of relations (for example, with the help of logical statements of any computer language) and quantities in computer files, the future role of mathematics in modeling will be greatly diminished.
However, the fact that mathematics is also the science of relations means that the role of mathematics in modeling is deeper than that. Thanks to this fact, it is possible to take an initial mathematical statement that merely describes other scientists' knowledge and transform it by a sequence of logically consistent operations to arrive at a new statement, or a model. The latter statement or model will be as true as the initial one but may exhibit a very new insight into the modeled phenomenon. In effect, by this process, the mathematical description acquires a status of a "theory," in the sense that all logical consequences of the initial description become available as tools to either support or reject the theory.

It is, perhaps, inevitable that as computer systems continue to evolve to imitate the processes of mathematical analysis and algebraic manipulation (as the many currently available symbolic manipulators already attempt), software packages will emerge that will—to varying degrees—automate the process of creating and transforming a mathematical model. We choose to call this new generation of models *mathematical computer models*, as distinguished from mathematical models merely implemented on a computer; and we shall discuss this issue in more detail later on.

In addition to the previously mentioned capability to reduce complexity of real phenomena by extracting only certain essential features that are of interest, some mathematical models also have the capability to idealize "imperfect" real phenomena. Perhaps one of the most trivial examples of this is the equation of a circle:

$$x^2 + y^2 = r^2$$

where r is the radius. While conceptually we may define a circle as a set of points equidistant from the centre point, we are incapable of constructing such an object or, indeed, of observing it in nature except to a certain degree of accuracy. And yet the equation constitutes a perfect, absolutely precise, description of a circle. In this case, if the equation is thought of as the model, then it is an idealized model, and all real world "circles" are, at best, approximations to such an "ideal" circle.

Up to this point we have attempted to point to certain powerful and generic features that mathematical modeling has to offer. However, perhaps the single most compelling rationale—though not independent of the preceding ones—for the use of mathematical models is historical. Mathematics is the universal language of science, engineering, and increasingly of other disciplines.

Much of the recorded knowledge of physics and engineering is written in the form of mathematical models. These mathematical models form the foundations of our understanding of the universe we live in. Furthermore, nearly all of the existing technology, in one way or another, rests on these models. To the extent that we are surrounded by evidence of the technology working and being reliable, human confidence in the validity of the underlying mathematical models is all but unshakable. Even when revolutions in physics such as Einstein's discovery of relativity take place, they merely re-emphasize that the old Newtonian models work exceedingly well in the parameter ranges in which we could normally wish to use them.

However, it is not only the long history of success of mathematical models in sciences that provides strong support for their continued use. The rationale for their continued and even much expanded role is stronger than ever because of the revolution in high-speed computing. The latter has been accompanied by a revolution in algorithms for numerically solving, at great speed, larger and more complex systems of equations, which in turn opens up the opportunity of modeling increasingly complex phenomena.

In recent years we have witnessed the emergence of very sophisticated models: both local short-term models of weather prediction, and coupled ocean–atmosphere models of global climate change. The numerical solution of the underlying systems of equations, considered impossible not long ago, has now become possible due to the combination of powerful computers and powerful algorithms. This combination of algorithmic and computational power is providing a tremendous impetus for a much expanded role of mathematical modeling, and brings with it many opportunities, as well as some dangers that will be discussed later on in this introduction and in many other places across the theme.

To summarize this section we observe that the following are among the key drivers providing the rationale for continued and even expanded role of mathematical modeling of life support systems:

- the dual nature of mathematics as the "science of relations" as well as the "science of magnitude"
- the inherent ability of a mathematical model to provide a "theory" describing the modeled phenomenon
- the inherent ability of mathematics to reduce complexity by modeling only a few of the most relevant characteristics of the problem
- the ability of mathematics to create idealized models that serve as benchmarks for physical entities
- the historical role of mathematics as the universal language of science and engineering
- the opportunities presented by the combined power of algorithms and high speed computing.

-
-
-

Bibliography

Basmadjian, D. 1999. *The Art of Modeling in Science and Engineering.* London, Chapman and Hall. [A textbook for advanced students and practicing professionals in the sciences and engineering with interest in modeling.]

Batchelor, G. K. 1994. *An Introduction to Fluid Dynamics.* Cambridge, Cambridge University Press. [A classic textbook on the fundamental theory of fluid mechanics.]

Beltrami, E. J. 1987. *Mathematics for Dynamic Modeling.* Boston, Academic Press. [An introduction to mathematical tools needed for the understanding of equilibrium, stability, limit cycles, bifurcation and chaos and other concepts.]

Bender, J. D. 1978. *An Introduction to Mathematical Modeling.* New York, John Wiley. [A textbook designed to teach students how to formulate, analyse and criticize mathematical models.]

Fielding, A. H. (ed.) 1999. *Machine Learning Methods for Ecological Applications.* Boston, Kluwer Academic. [A collection of papers dealing with the topic of machine learning methods in ecological modeling.]

Finch, P. D. 1981. On the role of description in statistical inquiry. *Philosophy of Science,* Vol. 32. [Contains an insightful discussion of the role of exploratory data analysis in statistics.]

Forrester, J. W. 1971. *World Dynamics.* Cambridge, Mass, Wright-Allen Press. [One of the classical, and much criticized, global models simulating the world's economic, social, political and environmental systems.]

Giordano, F. R.; Weir, M. D. 1985. *A First Course in Mathematical Modeling.* Monterey, Calif., Brooks/Cole. [A comprehensive introduction to mathematical modeling techniques in a wide range of contexts.]

Haefner, J. W. 1996. *Modeling Biological Systems: Principles and Applications.* New York, Chapman and Hall. [A textbook on modeling biological systems with emphasis on simulations.]

Haidvogel, D. B.; Beckmann, A. 1999. *Numerical Ocean Circulation Modeling.* Series on Environmental Science and Management. London, Imperial College Press. [A research-level presentation of mathematical techniques in the context of ocean circulation modeling.]

Hargrove, J. L. 1998. *Dynamic Modeling in the Health Sciences.* New York, Springer. [A book suitable for health science professionals presenting the use of dynamic modeling and simulation techniques in the context of complex life sciences systems.]

Holland, J. H. 1998. *Emergence: From Chaos to Order*. Reading, Mass., Helix Books, Addison-Wesley. [An easily accessible introduction to the concepts of emergence, complexity and agent based modeling.]

Huerlimann, A. 1999. Mathematical modeling and optimization. An essay for the design of computer-based modeling tools. *Applied Optimization*, Vol. 31. [An essay on mathematical modeling of optimization problems with the help of high-level modeling languages, in particular the Logical Programming Language (LPL).]

Isaacs, R. 1979. On applied mathematics. *Journal of Optimization Theory and Applications*, Vol. 27, No. 1, pp. 31–50. [Reflections by an accomplished applied mathematician about the distinctions between pure and applied mathematics.]

James, A. (ed.) 1978. *The Use of Mathematical Modeling in Water Pollution Control.* Chichester/New York, Wiley. [A collection of papers describing a range of mathematical models of water pollution, monitoring and treatment.]

Jorgensen, S. E. 1994. *Fundamentals of Ecological Modeling.* Amsterdam, Elsevier Science. [A handbook for engineers and ecologists providing an introduction to the field of ecological and environmental modeling.]

Kulkarni, V. G. 1999. *Modeling, Analysis, Design, and Control of Stochastic Systems.* New York, Springer. [A classical textbook on the modeling of stochastic systems.]

Malthus, T. R. 1983. *An Essay on the Principle of Population.* Harmondsworth, UK, Penguin. [An eighteenth-century classical and controversial essay on the limits of population growth.]

Mandelbrot, B. B. 1999. *Multifractals and 1/f Noise: Wild Self-Affinity in Physics (1963–1976). Selecta Volume.* New York, Springer. [A major contribution to the understanding of wild self-affine variability and randomness.]

May, R. M. 1976. Simple mathematical models with very complicated dynamics. *Nature*, No. 261, pp. 459–67. [A classical paper exposing the presence of chaos in some ecological models.]

Meerschaert, M. M. 1999. *Mathematical Modeling.* San Diego, Calif., Academic Press. [A practical overview of selected mathematical modeling techniques of optimization, dynamical systems, and stochastic processes with applications to a variety of real problems.]

Neuilly, M. 1999. *Modeling and Estimation of Measurement Errors.* Preface by C. Frejacque. Transl. from the French by D. J. Pike. Andover, Intercept. [A book dealing with statistical methods to improve the quality of analysis of measurable data.]

Rotmans, J. 1990. *IMAGE: An Integrated Model to Assess the Greenhouse Effect: Environment & Assessment.* Dordrecht, Kluwer Academic. [A detailed account of an integrated greenhouse effect model developed at RIVM in the Netherlands.]

Rotmans, J.; deVries, B. 1997. *Perspectives on Global Change: The TARGETS Approach.* Cambridge, Cambridge University Press. [A detailed description of an integrated assessment model known as the Tool to Assess Regional and Global Environmental and health Targets for Sustainability (TARGETS, for short).]

Saaty, T. L.; Alexander, J. M. 1981. *Thinking with Models: Mathematical Models in the Physical Biological and Social Sciences.* Oxford, Pergamon. [A textbook on mathematical modeling which emphasizes the model formulation phase.]

Sigmund, K. 1993. *Games of Life: Explorations in Ecology Evolution and Behaviour.* New York, Oxford University Press. [A popular exposition of a range of topics related to artificial life and self-replicating machines, including some topics from the theory of games, cellular automata, chaotic motion and its role in ecology.]

Thompson, J. R. 2000. *Simulation. A Modeler's Approach.* New York, Wiley. [A book on the statistical approach to simulation modeling, emphasizing simulation as an integral part of the modeling process.]

Tukey, J. W. 1977. *Exploratory Data Analysis.* Reading, Mass., Addison-Wesley. [A seminal treatise on exploratory data analysis.]

Von Neumann, J.; Morgenstern, O. 1953. *Theory of Games and Economic Behavior*. 2nd edn. Princeton, N.J., Princeton University Press. [A seminal treatise that formed a foundation of the subject of Game Theory.]

Wan, F. Y. M. 1989. *Mathematical Models and Their Analysis.* New York, Harper & Row. [An advanced undergraduate textbook on mathematical modeling.]

Biographical Sketch

Jerzy A. Filar, is a broadly trained applied mathematician with research interests spanning both theoretical and applied topics in operations research, optimization, game theory, applied probability, and environmental modeling. After sixteen years of academic career in the United States, which included appointments at the University of Minnesota, Johns Hopkins University and the University of Maryland, he returned to Australia in 1992 where he holds the Foundation Chair of Mathematics and Statistics at the University of South Australia. He has co-authored (with K. Vrieze) a research level text book *Competitive Markov Decision Processes* (Springer, 1996). He also authored or co-authored approximately 70 refereed research papers which appeared in a wide spectrum of international journals ranging from theoretical mathematics journals such as *Proceedings of the American Mathematics Society*, through most of the mainstream optimization/OR journals, to very applied journals such as *Applied Mathematical Modelling, Socio-Economic Planning,* and *Energy Economics*.He has worked on applications of mathematical modeling in a number of different contexts. These included the issues of error propagation in integrated models of enhanced greenhouse effect, the issues of efficient scheduling of inspections, optimization of recovery from disruptions at airports, the use of data in litigation, and the problems associated with aggregation of various measures of performance. Some of these research projects were sponsored by government agencies and research institutes such as the US EPA, the World Resources Institute, and the Sir Keith and Sir Ross Smith Foundation in South Australia. He is editor-in-chief of *Environmental Modeling and Assessment* and an Associate Editor of the *Journal of Mathematical Analysis and Applications, Mathematical Methods of Operations Research* (*Zeitschrift fur OR*) and *International Game Theory Reviews*. He has held short-term posts as Visiting Professor at the Universities of Maastricht, Toulouse, Ulm, the Technical University of Vienna and the Chinese University of Hong Kong. In 1994 he founded CIAM (the Center for Industrial and Applicable Mathematics) at the University of South Australia. During his tenure as the director (which ended in 1998) this center has grown into one of the leading centers of its type in Australia and has also established an international profile.