

STABILITY CONCEPTS

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Summary

Stability plays a very important role in system theory and control design. The most fundamental concepts of stability were introduced by A.M. Lyapunov in the late 19th century. Lyapunov not only gave a formal statement of the problem but also proposed the methods which till today serve as key instruments for treating the stability problems. Originally developed for a family of motions defined for ordinary differential equations, the Lyapunov stability concepts were lately applied to dynamical systems in more abstract spaces and even to general motions which are not described by the equations studied in classical analysis. Subsequently, Lyapunov's concepts were also adopted to

investigate more complicated phenomena in the behavior of dynamical systems such as bifurcation and chaos. The results of the stability theory have applications in examining motion in space, technological devices, automated systems, problems in mechanics, environmental studies, economics and behavioral science and many others.

1. The Definition of Stability

1.1. Introduction

The notion of stability is as old as the civilized world and has a very clear intuitive meaning. Take an ordinary pendulum and put it in the lowest position, in which it is “stable”. Put it in the utmost upper position where it is “unstable”. Stable and unstable situations can be met everywhere – in mechanical motion, in technical devices, in medical treatment (stable or unstable state of the patient), in currency exchange and so on. The rigorous mathematical theory of stability had appeared in the course of studying mechanical motions with some early definitions of stability given by *Joseph L. Lagrange* (for example, a “stable” position for a pendulum is when its potential energy attains a minimum). Another definition was introduced later by *Siméon D. Poisson*, followed by others.

Perhaps the most widely known *theory of stability of motion* well applicable to engineering and many other applied problems is due to *Alexander M. Lyapunov*. (Alexander Michailovich Lyapunov, (1857-1918) – a distinguished Russian mathematician famous for his work on stability theory and problems in probability. Member of the Russian Academy of Sciences, professor of the Kharkov University and later of the St. Petersburg University. Lyapunov’s concepts and methods are widely used in the mathematical and engineering communities.) The notions of Lyapunov stability and asymptotic stability are followed by those of exponential stability, conditional stability, stability over a part of the variables, stability under persistent disturbances and others. In terms of such notions many natural phenomena were explained (as in astronomy, for example).

They are also widely used in engineering design, where modified notions of “stochastic stability”, “absolute stability” and others had been introduced. A crucial point in Lyapunov’s theory is the introduction of so-called “Lyapunov functions” the knowledge of which allows us to identify stable systems (described by ordinary differential equations, for example), without solving (“integrating”) them. Lyapunov’s methods and their further developments are widely used in *control theory* and automated control design, where such notions as “input-output stability” and “control Lyapunov functions” have appeared.

Beyond the scope of the present chapter also are problems of stability for distributed parameter processes such as hydrodynamic stability, stability of elasto-plastic and deformable systems, stability of bodies with cavities containing liquid etc. Subsequently Lyapunov’s concepts were also used to study more complicated phenomena, such as bifurcation, chaos and turbulence. There are also attempts to use these concepts in mathematical models for economics, demography, biomedical problems and other applied areas.

1.2. The Concept of Lyapunov's Stability

Consider a dynamic process described by a system of ordinary differential equations written in a normal form

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_n, t), \quad i = 1, \dots, n,$$

where $\dot{x}_i = dx_i / dt$. Here t is an independent variable which usually denotes time. The vector $(x_1(t), \dots, x_n(t))^T = \mathbf{x}(t) \in \mathbf{R}^n$ is the state vector, and \mathbf{R}^n is the state space. If each f_i are independent of t , the system is called *autonomous*, or *time-invariant*.

This system can be written in vector form as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t). \quad (1)$$

Given a vector $\mathbf{x}_0 \in \mathbf{R}^n$ and a time instant t_0 , consider the problem to find a solution to (1) satisfying the *initial condition* $\mathbf{x}(t_0) = \mathbf{x}_0$. This problem is called an *initial-value problem* for (1).

Suppose that each f_i is continuous and has continuous partial derivatives with respect to each of the x_1, \dots, x_n in an open domain $\{\mathbf{x} \in \Omega, \underline{t} < t < \infty\}$. Then, for every $\mathbf{x}_0 \in \Omega$ and $t_0 > \underline{t}$, there exists a unique solution $\mathbf{x}(t)$ to the initial-value problem $\mathbf{x}(t_0) = \mathbf{x}_0$ for (1) and this solution is defined on an open time interval containing t_0 .

Let $\bar{\mathbf{x}}(t)$ be a particular solution of Eq.(1), which is extendable throughout the semiaxis $[t_0, +\infty)$ and whose stability properties one has to study. Following Lyapunov's terminology, this solution is referred to as the *unperturbed motion*, whereas all the others are said to be *perturbed*. After shifting the variables as $\mathbf{x}'(t) = \mathbf{x}(t) - \bar{\mathbf{x}}(t)$, one may agree that $\bar{\mathbf{x}}(t) \equiv \mathbf{0}$ and $\mathbf{f}(\mathbf{0}, t) = \mathbf{0}$ for all $t \geq t_0$. Let $\|\cdot\|$ stand for the Euclidean norm for a vector in \mathbf{R}^n .

Definition 1 A state $\mathbf{x} = \mathbf{c}$ is said to be an equilibrium state of the system (1) if $\mathbf{f}(\mathbf{c}, t) \equiv \mathbf{0}$ for all $t \geq t_0$.

In Lyapunov stability theory, the behavior of the perturbed motions whose initial state \mathbf{x}_0 is in a small neighborhood of the equilibrium state $\mathbf{x} = \mathbf{0}$ is studied. The following core definition is due to Lyapunov (1892).

Definition 2 The equilibrium state $\mathbf{x} = \mathbf{0}$ is called

- Lyapunov stable if for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for any \mathbf{x}_0 , the perturbed motion issuing from \mathbf{x}_0 at $t = t_0$, is extendable throughout the

semiaxis $[t_0, \infty)$ and yields $\|\mathbf{x}(t)\| < \varepsilon$ for all $t \geq t_0$.

- asymptotically stable if it is stable and, in addition, there exists a $\delta > 0$ such that for each perturbed motion satisfying $\|\mathbf{x}(t_0)\| < \delta$, one has

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}. \quad (2)$$

- unstable if it is not stable.

Later these properties appeared in a general form. More properties were invented.

Definition 3 The equilibrium state $\mathbf{x} = \mathbf{0}$ is called exponentially stable if there exist three positive numbers δ_0, η, c such that $\|\mathbf{x}(t)\| \leq \eta \|\mathbf{x}(t_0)\| e^{-c(t-t_0)}$ holds for every perturbed motion with $\|\mathbf{x}(t_0)\| < \delta_0$.

Definition 4 The equilibrium state $\mathbf{x} = \mathbf{0}$ is called

- asymptotically stable in the large, or completely stable, or stable in the whole, if it is stable and $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$ for each perturbed motion $\mathbf{x}(t)$, and,
- exponentially stable in the large if there exist positive numbers η, c such that $\|\mathbf{x}(t)\| \leq \eta \|\mathbf{x}(t_0)\| e^{-c(t-t_0)}$ holds for every perturbed motion.

Definition 5 The equilibrium state $\mathbf{x} = \mathbf{0}$ is called

- uniformly stable (Persidskii, 1933) if for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that, for any $\tau \geq t_0$, the inequality $\|\mathbf{x}(\tau)\| < \delta$ implies $\|\mathbf{x}(t)\| < \varepsilon$ for all $t \geq \tau$,
- uniformly asymptotically stable (Malkin, 1954) if it is uniformly stable and there is a $\delta_0 > 0$ with the following property: for each $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that $\|\mathbf{x}(\tau)\| < \delta_0$ implies $\|\mathbf{x}(t)\| < \varepsilon$ for all $t \geq \tau + T$, and
- uniformly asymptotically stable in the large if it is uniformly stable, and for each $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that $\|\mathbf{x}(t)\| < \varepsilon$ whenever $t \geq T + t_0$.

We note that the stability property implies the perturbed motions with $\|\mathbf{x}(\tau)\| < \delta$ to be extendable throughout the semi axis $[\tau, +\infty)$. The sufficient condition for it is that, for example, that, in addition to the continuity of all f_i and $\partial f_i / \partial x_j, i, j = 1, 2, \dots, n$, the inequality $z |f_i(x_1, x_2, \dots, x_n, t)| \leq \kappa(t) \|\mathbf{x}\|$ holds where $\kappa(t)$ is a continuous function. .

There are known other concepts than Lyapunov's stability which are used to qualify the behavior of perturbed motions (see also Section 1.9 of this article). For example, the equilibrium $\mathbf{x} = \mathbf{0}$ for Eq.(1) is said to be *attractive* if there exists an open domain

$\Upsilon \subseteq \mathbf{R}^n$ containing the equilibrium such that $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$ whenever $\mathbf{x}(\tau) \in \Upsilon$, there with

the domain Υ is called the *domain of attraction*. Thus, the equilibrium is asymptotically stable if and only if (iff) it is stable and attractive. To be sure, the properties *unstable* and *attractive* are not mutually exclusive even for an autonomous system, as shows the following example by Vinograd.

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}, \quad \dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}, \quad (3)$$

In (3), the right sides are defined to be zero for $x_1 = x_2 = 0$. In this system, the origin is unstable but attractive. Some methods are available for estimating the domain of attraction.

Also, it should not be taken that exponential stability of a solution implies its stability in the sense of Lyapunov, as Perron's example shows:

$$\dot{x}_1 = -ax_1, \quad \dot{x}_2 = (\sin(\ln t) + \cos(\ln t) - 2a)x_2 + x_2^2$$

Here, the null solution is exponentially stable, but if $a \in (\frac{1}{2}, \frac{2+e^{-\pi}}{4})$, it is not stable.

1.3. The Second (Direct) Method of Lyapunov

The main qualitative method for investigating stability properties of an unperturbed motion is the *direct method of Lyapunov* also known as the *second method of Lyapunov*. The aim of the method is to reduce the system stability analysis to the analysis of the properties of some special "Lyapunov" functions, presuming that this could be done *without integrating the original system*.

Consider a function $V(\mathbf{x}, t)$ which is continuous and has continuous partial derivatives with respect to each of the arguments x_1, x_2, \dots, x_n, t in a domain $Z = \{\|\mathbf{x}\| < h, t_0 \leq t < \infty\}$. Some special terms are frequently used.

Definition 6 (i) *The function $V(\mathbf{x}, t)$ is called*

- positive (negative) semi-definite in Z if $V(\mathbf{x}, t) \geq 0$ (respectively, $V(\mathbf{x}, t) \leq 0$) for all $(\mathbf{x}, t) \in Z$
- positive definite in Z if there exists a function $w(r)$, which is continuous and strictly increasing in $r \in [0, h)$, and $w(0) = 0$ and such that

$$V(\mathbf{x}, t) \geq w(\|\mathbf{x}\|) \quad (4)$$

for all $(\mathbf{x}, t) \in Z$.

- negative definite if $-V(\mathbf{x},t)$ is positive definite.
- decrescent if there exists a continuous strictly increasing function $\varphi(r), r \in [0, h)$ such that $\varphi(0) = 0$ and $V(\mathbf{x},t) \leq \varphi(\|\mathbf{x}\|)$ in Z .
- radially unbounded if V is defined in $\{\mathbf{x} \in \mathbf{R}^n, t \geq t_0\}$, it is positive definite and there is a function $w(r), r \geq 0$ yielding (4) and $\lim_{r \rightarrow \infty} w(r) = \infty$.

(iv) (ii) *The derivative*

$$\dot{V}(\mathbf{x},t) \equiv \frac{\partial V(\mathbf{x},t)}{\partial t} + \sum_{i=1}^n \frac{\partial V(\mathbf{x},t)}{\partial x_i} f_i(\mathbf{x},t) \quad (5)$$

is called the derivative of V along a motion of Eq.(1).

Theorem 1 (The first theorem of Lyapunov, or Lyapunov’s theorem on stability)

If for Eq.(1) there exists a positive definite function $V(\mathbf{x},t)$ with a negative semi-definite derivative $\dot{V}(\mathbf{x},t)$, then the equilibrium state $\mathbf{x} = \mathbf{0}$ of this equation is Lyapunov stable.

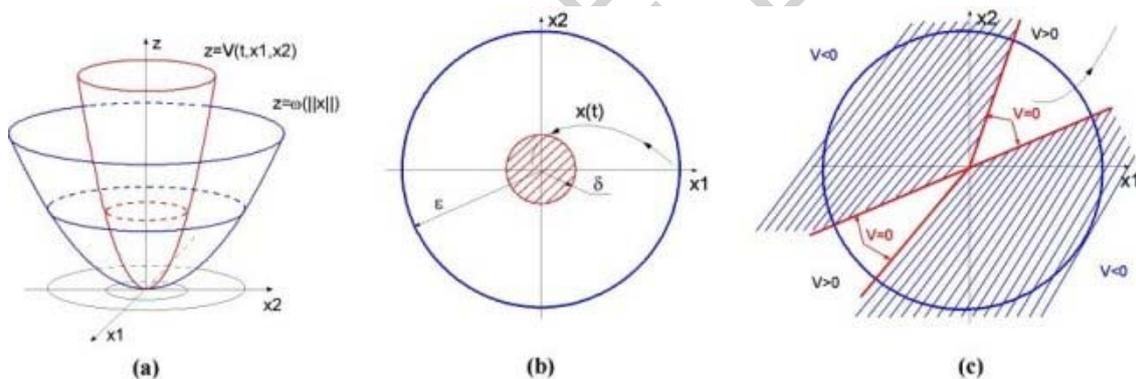


Figure 1: (a) A Lyapunov function. (b) An illustration to Lyapunov’s theorem on asymptotic stability. (c) An illustration to Chetayev’s theorem.

Theorem 2 (The second theorem of Lyapunov, or Lyapunov’s theorem on asymptotic stability) The equilibrium state $\mathbf{x} = \mathbf{0}$ of Eq.(1) is asymptotically stable as $t \rightarrow \infty$ if there exists a positive definite decrescent function $V(\mathbf{x},t)$ with a negative definite derivative $\dot{V}(\mathbf{x},t)$.

Theorem 3 (The first theorem of Lyapunov on instability) The equilibrium state $\mathbf{x} = \mathbf{0}$ of Eq.(1) is unstable if there exists a function $V(\mathbf{x},t)$ in a domain $Z = \{\|\mathbf{x}\| < h, t_0 \leq t < \infty\}$ such that

- (i) $V(\mathbf{x},t)$ is decrescent in Z ;
- (ii) $\dot{V}(\mathbf{x},t)$ is positive definite in Z ;

- (iii) *there exists $\hat{t} > t_0$ such that for any $\varepsilon \in (0, h)$, there exists $\mathbf{x}, \|\mathbf{x}\| < \varepsilon$, for which $V(\mathbf{x}, \hat{t})\dot{V}(\mathbf{x}, \hat{t}) > 0$.*

Theorem 4 (The second theorem of Lyapunov on instability) *Let there exist a bounded function $V(\mathbf{x}, t)$ in the domain $Z = \{\|\mathbf{x}\| < h, t_0 \leq t < \infty\}$ with the following properties:*

- (i) *$\dot{V}(\mathbf{x}, t) = gV + W(\mathbf{x}, t)$ where g is a positive constant and $W(\mathbf{x}, t)$ is either identically zero or semi-definite;*
 (ii) *in case $W(\mathbf{x}, t)$ is not identically zero, in each domain $Z_1 = \{\|\mathbf{x}\| < h_1, t_1 \leq t < \infty\}$ with arbitrarily large t_1 and arbitrarily small h_1 , there exists an \mathbf{x} such that $V(\mathbf{x}, t)$ and $W(\mathbf{x}, t)$ have the same sign for $t \geq t_1$. Then the equilibrium is unstable.*

Theorem 5 (Chetayev's theorem on instability) *The equilibrium state $\mathbf{x} = \mathbf{0}$ of Eq.(1) is unstable if there exists a function $V(\mathbf{x}, t)$ in a domain $Z = \{\|\mathbf{x}\| < h, t_0 \leq t < \infty\}$ such that*

- (i) *for any $t \geq t_0$, there exists a nonempty domain $Z_+(t) \subset \{\mathbf{x} : \|\mathbf{x}\| < h\}$ where $V(\mathbf{x}, t) > 0$ for each $\mathbf{x} \in Z_+(t)$ and $\mathbf{x} = \mathbf{0}$ is a boundary point of $Z_+(t)$;*
 (ii) *for any $t \geq t_0$, $V(\mathbf{x}, t)$ is bounded and $\dot{V}(\mathbf{x}, t) > 0$ in $Z_+(t)$;*
 (iii) *for any $\alpha > 0$ there exists a $\beta = \beta(\alpha) > 0$ such that from $V(\mathbf{x}, t) > \alpha$ it follows that $\dot{V}(\mathbf{x}, t) > \beta$.*

In Figure 1(c), the trajectory $\mathbf{x}(t)$ is shown, for which $V(\mathbf{x}_0, t_0) > \alpha > 0$. Since $\dot{V}(\mathbf{x}, t) > \beta(\alpha) > 0$, $V(\mathbf{x}(t), t)$ is infinitely increasing. As a consequence, the trajectory $\mathbf{x}(t)$ has eventually to leave any neighborhood of the origin.

Theorem 6 (on exponential stability) *The equilibrium $\mathbf{x} = \mathbf{0}$ of system (1) is exponentially stable if there exists an $\varepsilon > 0$ and a function $V(\mathbf{x}, t)$ which satisfies*

$$\alpha_1 \|\mathbf{x}\|^2 \leq V(\mathbf{x}, t) \leq \alpha_2 \|\mathbf{x}\|^2$$

$$\dot{V}(\mathbf{x}, t) \leq -\alpha_3 \|\mathbf{x}\|^2$$

$$\left\| \frac{\partial V(\mathbf{x}, t)}{\partial x_i} \right\| \leq \alpha_4 \|\mathbf{x}\|$$

for some positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and all $\|\mathbf{x}\| \leq \varepsilon, t \geq t_0$.

Originally, Theorem 6 was formulated by Krasovskii in 1959 for time-invariant systems

supposed that $V(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ and $\dot{V} \leq -\mathbf{x}^T \mathbf{B} \mathbf{x}$ for positive definite symmetric matrices \mathbf{A} and \mathbf{B} .

Now consider an autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (6)$$

which has the equilibrium state $\mathbf{x} = \mathbf{0}$.

Theorem 7 (Barbashin-Krasovskii's theorem on the asymptotic stability in the large) For system (6) let there exist a radially unbounded function $V(\mathbf{x})$ and a set $M \subseteq \mathbf{R}^n$ such that

- (i) $\dot{V}(\mathbf{x}) < 0$ if $\mathbf{x} \in \mathbf{R}^n \setminus M$ and $\dot{V}(\mathbf{x}) \leq 0$ if $\mathbf{x} \in M$;
- (ii) whatever the positive constant c is, there exist no semi-trajectory $\mathbf{x}(t), t \geq 0$ of system (6) that lies in an intersection of M and the set $V(\mathbf{x}) = c$.

Then the equilibrium state $\mathbf{x} = \mathbf{0}$ is asymptotically stable in the large.

Barbashin-Krasovskii's theorem enables one to conclude asymptotic stability in the large even when the derivative of V along a motion of (6) is not positive definite. From Theorem 7, La Salle's principle follows, which is formulated in terms of an *invariant set*. The set $M \subseteq \mathbf{R}^n$ is said to be an invariant set for (6) if for all $\mathbf{x}_0 \in M$ and all $t_0 \geq 0$, the inclusion $\mathbf{x}(t) \in M$ holds whenever $\mathbf{x}(t_0) = \mathbf{x}_0$.

Theorem 8 (La Salles's Principle) For Eq.(6), let there exist a positive definite function $V(x)$ such that on the compact set $Z = \{\mathbf{x} \in \mathbf{R}^n : V(\mathbf{x}) \leq c\}$ one has $\dot{V}(\mathbf{x}) \leq 0$. Define $S = \{\mathbf{x} \in Z : \dot{V}(\mathbf{x}) = 0\}$. If S contains no invariant set other than $\mathbf{x} = \mathbf{0}$, then the origin is asymptotically stable.

Theorem 9 (Krasovskii's theorem on asymptotic stability in the large) For system (6) let the Jacobian matrix $\mathbf{J}(\mathbf{x}) = \{(\partial f_i(\mathbf{x}) / \partial x_j)\}$ satisfy the inequality

$$\mathbf{J}(\mathbf{x}) + \mathbf{J}^T(\mathbf{x}) \leq -\epsilon \mathbf{I} < 0, \epsilon > 0$$

where \mathbf{I} is the identity matrix of order n . Then the equilibrium state $\mathbf{x} = \mathbf{0}$ is asymptotically stable in the large and $V(\mathbf{x}) = \|\mathbf{f}(\mathbf{x})\|^2$ is a Lyapunov function for (6).

Theorem 10 (Persidskii's theorem on uniform stability) If for Eq.(6), there exists a positive definite decrescent function $V(\mathbf{x}, t)$ with a negative definite derivative $\dot{V}(\mathbf{x}, t)$, then the equilibrium state $\mathbf{x} = \mathbf{0}$ of this equations uniformly stable.

Theorem 11 (Malkin's theorem on uniform asymptotic stability) If for Eq.(6), there

exists a positive definite decrescent function $V(\mathbf{x},t)$ with a negative definite derivative $\dot{V}(\mathbf{x},t)$, then the equilibrium state $\mathbf{x} = \mathbf{0}$ of this equation is uniformly asymptotically stable.

Theorem 12 (on uniform asymptotic stability in the large) *Let the motions of Eq.(6) be defined in the entire space \mathbf{R}^n . Let there exist a radially unbounded function $V(\mathbf{x},t)$ satisfying the hypothesis of Theorem 1 in the domain $Z = \{\mathbf{x} \in \mathbf{R}^n : t \geq t_0\}$. Then the equilibrium state $\mathbf{x} = \mathbf{0}$ of this equation is uniformly asymptotically stable in the large.*

Theorems listed above furnish *sufficient conditions* for stability or instability but say nothing about how a suitable Lyapunov function can be found. It is remarkable, however, that some of these theorems can be *conversed*, i.e. from known stability properties, the existence of suitable Lyapunov function may be inferred. For example, for an asymptotically stable linear time-invariant system, a Lyapunov function ensuring asymptotic stability can always be found in the class of quadratic forms (see Section 1.5 in this chapter).

An important way to generalize the Lyapunov second method is to introduce nonsmooth Lyapunov functions.

1.4. Sylvester's Criterion

For a broad class of differential equations, a Lyapunov function can be looked for in the class of quadratic forms

$$V(\mathbf{x}) = \sum_{i,j=1}^n \alpha_{ij} x_i x_j. \quad (7)$$

This leads to a fairly simple criterion for positiveness of a quadratic form to be of use. The *Sylvester criterion* gives necessary and sufficient conditions for a quadratic form (7) with real coefficients α_{ij} to be a positive definite function. Assume that the form (7) is symmetric, i.e. $\alpha_{ij} = \alpha_{ji}$ for all i, j . Then one may write $V(\mathbf{x}) = \mathbf{x}^T \mathbf{\Lambda} \mathbf{x}$, where the $n \times n$ matrix $\mathbf{\Lambda}$ has the entries α_{ij} .

Theorem 13 (Sylvester's criterion) *For a symmetric quadratic form (7) with real coefficients to be positive definite, it is necessary and sufficient that all the principal sub-determinants of the matrix $\mathbf{\Lambda}$ be positive, i.e.*

$$\alpha_{11} > 0, \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} > 0, \dots, \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix} > 0$$

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