CONTROL OF NONLINEAR SYSTEMS

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Contents

Summary

The last quarter of the twentieth century has seen a rapid progress towards the development of a nonlinear control theory. This chapter introduces the main tools of analysis and design of nonlinear control systems, which are detailed in the subsequent chapters.

1. Introduction

There are many control tasks that require the use of feedback. Depending on the design goals, there are several formulations of the control problem. The tasks of stabilization, tracking, and disturbance rejection or attenuation (and various combinations of them) lead to a number of control problems. In each one, we may have a state feedback version where all state variables can be measured, or an output feedback version where all state variables can be measured, or an output feedback version where only an output vector, whose dimension is often less than the dimension of the state, can be measured. In a typical control problem, there are additional goals for the design, like meeting certain specifications on the transient response or certain constraints on the control input. These requirements could be conflicting and the designer has to trade them off. The desire to optimize the design leads to various optimal control formulations. When model uncertainty is taken into consideration, issues of sensitivity and robustness come into play. The attempt to design feedback control to cope with a wide range of model uncertainty leads to either robust or adaptive control problems. In robust control, the model uncertainty is characterized as a perturbation of a nominal model. You may think of the nominal model as a point in a space and the perturbed models as points in a ball that contains the nominal model. A robust control design tries to meet the control objective for any model in the "ball of uncertainty." Adaptive control, on the other hand, parameterizes the uncertainty in terms of certain unknown parameters and tries to use feedback to learn these parameters on-line, that is, during the operation of the system. In a more elaborate adaptive scheme, the controller might be learning certain unknown nonlinear functions, rather than just learning some unknown parameters. There are also problem formulations that mix robust and adaptive control. In the current chapter, we limit our discussions to the basic tasks of stabilization, tracking, and disturbance rejection.

2. Stability

Stability analysis plays a central role in control. There are different concepts of stability. The most dominant one is the concept formulated by Lyapunov at the end of the nineteenth century and further developed by many researchers throughout the twentieth century. The concept is concerned with the stability of steady-state solutions, such as equilibrium points and periodic orbits. In the 1960s and 70s, concept of input-output stability and passivity were formulated by Popov, Sandberg, Willems, and Zames, among others. These concepts are particularly useful when we analyze the feedback connection of Figure 1.

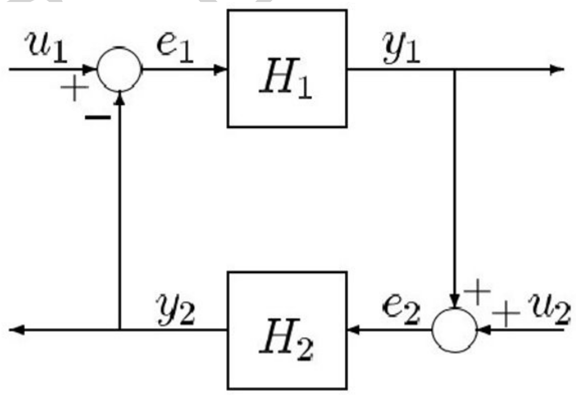

Figure 1: Feedback connection.

2.1. Lyapunov Stability

In its simplest form, Lyapunov stability is concerned with the stability of an equilibrium point of the nonlinear system $\dot{x} = f(x)$. Lyapunov formulated the notions of stability and asymptotic stability of an equilibrium point. For an asymptotically stable equilibrium point, all trajectories starting in a region that contains the point, called the region of attraction, converge to it as time tends to infinity. The key idea of Lyapunov stability is that if we take a scalar function $V(x)$ that vanishes at the equilibrium point and is positive in its neighborhood and if we calculate the time derivative of this function using the chain rule

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x)$$

then the sign of $\dot{V}$ reveals whether V is increasing or decreasing along the trajectory passing through x. If we can show that $\dot{V}$ is negative in the neighborhood of the equilibrium point, we can conclude that it is asymptotically stable. The ingenious idea here is that we do not have to solve the state equation in order to determine that the trajectories move toward the equilibrium point. We only need to examine the sign of $\dot{V}$. The chapter on *Lyapunov stability* reviews the basic elements of the theory. It also describes an important extension of the basic theory, known as the invariance principle, which allows us to relax the requirement that $\dot{V}$ be negative everywhere around the equilibrium point. It allows $\dot{V}$ to be zero in certain sets, as long as the trajectories cannot stay in those sets over a period of time. Lyapunov theory is very versatile and applies to a wide range of mathematical models. The main challenge is finding the Lyapunov function. Although there is no general systematic method to find a Lyapunov function, research over the years has shown how to choose Lyapunov function candidates for certain classes of nonlinear systems.

2.2. Input-Output Stability

While Lyapunov stability is developed for state models (simultaneous first-order differential equations), an alternative approach for modeling dynamical systems is the input-output approach. An input-output model relates the output of the system directly to the input, with no knowledge of the internal structure that is represented by the state equation. The system can be represented by the relation $y = Hu$, where H is some mapping or operator that specifies the output y in terms of the input u. For linear time-invariant systems, such input-output models take the form of the convolution integral, or its equivalent transfer function model. Developing similar models for nonlinear systems was challenging, but starting in the 1960s progress was made towards developing functional models for nonlinear systems. This is reviewed in the chapter on *Volterra and Fliess Series Expansion*.

In input-output stability, the input u belongs to a space of signals $\mathcal{L}$; e.g., the space of bounded signals or the space of square-integrable signals. Keeping aside some technicalities, we can say that the system is $\mathcal{L}$ stable if the output satisfies

$$\|y\| \le \gamma\left(\|u\|\right) + \beta$$

where $\|\cdot\|$ is an appropriately defined norm on the space signals, γ is a gain function, which is strictly increasing and vanishes at zero, and β is a nonnegative bias constant. When the preceding inequality takes the special form

$$\|y\| \le \gamma \|u\| + \beta$$

where γ is a positive constant, the system is finite-gain $\mathcal{L}$ stable and the smallest such γ is called the gain of the system. This notion of input-output stability is introduced in the chapter on *Input-Output Stability*. It applies, of course, to the case when the input-output relationship is determined by a state model, but its real strength comes from the fact that it applies to systems that cannot be represented by a finite-dimensional state model, such as time-delay and infinite-dimensional systems.

2.3. Passivity

In the study of physical systems, such as electrical networks or mechanical systems, the concept of stored energy is often useful in understanding the behavior of the system. For example, in an RLC electrical network with passive components, the energy absorbed by the network over any period of time is greater than or equal to the increase in the stored energy over the same period. In the 1960s, Popov, Zames, and others, and later on in the 1970s, Willems, Hill, Moylan, and others, were able to extend this notion to a dynamical system for which a physical energy might not be well defined. The extension is based on a storage function (playing the role of energy) and a supply rate (playing the role of power flow into a network) such that the integral of the supply rate over any period of time is greater than or equal to the increase in the storage function over the same period. Such a system is called dissipative. When the supply rate is the inner product of the input and output vectors, that is, $u^T y,$ the system is said to be passive. Passive systems are introduced in the chapters on *Analysis of Nonlinear Control Systems* and *Passivity Based Control*.

2.4. Feedback Systems

Input-output stability and passivity concepts have been very effective in analyzing the stability of the feedback connection of Figure 1. Two celebrated results for this system are the small-gain theorem and the passivity theorem (see *Analysis of Nonlinear Control Systems*, *Input-Output Stability*, *and Passivity Based Control*). The (classical) small-gain theorem says that if the feedback components H_1 and H_2 are finite-gain $\mathcal{L}$ stable with gains γ_1 and γ_2, then the feedback connection is finite-gain $\mathcal{L}$ stable if $\gamma_1\gamma_2 < 1$. The passivity theorem says that the feedback connection of two passive systems is passive. These two theorems can be viewed as nonlinear generalizations of the linear gain and phase results in the Nyquist-Bode theory. When H_1 and H_2 are stable linear time-invariant systems represented by their transfer functions, the Nyquist-Bode theory tells us that the feedback connection will be stable if the loop gain is less than one or the loop phase shift is less than 180°. The connection to the small-gain theorem is obvious. The connection to the passivity theorem can be seen by noting that for a linear system to be passive, its phase shift cannot exceed 90°.

The passivity theorem plays an important role in passivity based control (see *Passivity Based Control*). The small-gain theorem provides a conceptual framework for understanding many of the robustness results that arise in the study of dynamical systems, especially when feedback is used. Quite often, dynamical systems subject to model uncertainties can be represented in the form of a feedback connection with H_1, say, as a stable nominal system and H_2 as a stable perturbation. Then, the requirement $\gamma_1\gamma_2 < 1$ is satisfied whenever γ_2 is small enough. The classical small-gain theorem applies to finite-gain stability. In the 1990, Hill, Jiang, Mareels, Praly, and Teel extended the small-gain theorem to the more general case when a gain γ is replaced by a gain function $\gamma(\cdot)$, leading to a small-gain condition of the form

$$\gamma_1(\gamma_2(s)) < s, \qquad \text{for all } s > 0$$

3. Sensitivity Analysis and Asymptotic Methods

The chapter on *Analysis of Nonlinear Control Systems* describes some useful analysis tools, namely, sensitivity analysis, the averaging method and the singular perturbation method.

An essential factor in the validity of any mathematical model is the continuous dependence of its solutions on the data of the problem. Sensitivity equations describe the effect of small parameter variations on the performance of the system.

Exact closed-form analytic solutions of nonlinear differential equations are possible only for a limited number of special classes of differential equations. In general, we have to resort to approximate solutions. There are two distinct categories of approximation methods that engineers and scientists should have at their disposal as they analyze nonlinear systems: (1) numerical solution methods and (2) asymptotic methods. Asymptotic methods reveal multiple-time-scale structures inherent in many practical problems. Quite often, the solution of the state equations exhibits the phenomenon that some variables move in time faster than other variables, leading to the classification of variables as “slow” and “fast.” The averaging and singular perturbation methods deal with the interaction of slow and fast variables. In the case of averaging, the fast variables take the form of fast oscillations, while in singular perturbations they appear as rapidly decaying signals.

4. Linearization and Gain Scheduling

Faced with the difficult task of designing feedback control for nonlinear systems, it is only natural that control engineers appealed to the neat results available for linear systems. By linearizing a nonlinear system about an operating (equilibrium) point, or a desired trajectory, we obtain a linear model that approximates the nonlinear system in the vicinity of the operating point. We can then use the linear control theory to design a feedback controller, which we apply to the nonlinear system and expect it to work as long as the trajectories of the nonlinear system remain in the vicinity of the operating point. We illustrate the design-via-linearization approach by considering the stabilization problem. Consider the system

$$\dot{x} = f(x,u), \quad y = h(x) \tag{1}$$

where $f(0,0) = 0, h(0) = 0,$ and $f(x,u), h(x)$ are continuously differentiable in a domain that contains the origin $(x = 0, u = 0)$. We want to design an output feedback controller (using only measurements of y) to stabilize the system; that is, to make the origin an asymptotically stable equilibrium point of the closed-loop system. Linearization of (1) about $(x = 0, u = 0)$ results in the linear system

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{2}$$

where

$$A = \frac{\partial f}{\partial x}(x,u)\bigg|_{x=0,u=0}, \quad B = \frac{\partial f}{\partial u}(x,u)\bigg|_{x=0,u=0}, \quad C = \frac{\partial h}{\partial x}(x)\bigg|_{x=0}$$

and $[\partial f / \partial x]$ is a Jacobian matrix whose (i, j) element is $\partial f_i / \partial x_j$. Assume (A, B) is stabilizable and (A, C) is detectable and design a linear dynamic output feedback controller

$$\dot{z} = Fz + Gy, \quad u = Lz + My \tag{3}$$

such that the closed-loop matrix

$$\begin{bmatrix} A + BMC & BL \\ GC & F \end{bmatrix} \tag{4}$$

is Hurwitz; that is, all its eigenvalues have negative real parts. An example of such design is the observer-based controller

$$\dot{z} = (A - BK - HC)z + Hy, \quad u = -Kz \tag{5}$$

where K and H are designed such that $A - BK$ and $A - HC$ are Hurwitz. The details of such a linear design are given in the chapters on *Design of State Space controllers (Pole Placement) for SISO systems* and *Control of Linear Multivariable Systems*. When the controller (3) is applied to the nonlinear system (1) it results in the closed-loop system

$$\dot{x} = f(x, Lz + Mh(x)), \quad \dot{z} = Fz + Gh(x) \tag{6}$$

It can be verified that the origin $(x = 0, z = 0)$ is an equilibrium point of the closed-loop system (6) and linearization about the origin results in the Hurwitz matrix of (4). It follows from Lyapunov theory that the closed-loop system (6) has an asymptotically stable equilibrium point at the origin.

The linearization approach is clearly local; that is, it can only guarantee asymptotic stability but it cannot, in general, prescribe the region of attraction nor achieve global asymptotic stability. Gain scheduling is a technique that can extend the validity of the linearization approach to a range of operating points. In many situations, it is known how the dynamics of a system change with its operating point. It might even be possible to model the system in such a way that the operating point is parameterized by one or more variables, which are called scheduling variables. In such situations, we may linearize the system at several equilibrium points (corresponding to different values of the scheduling variables), design a linear feedback controller at each point, and implement the resulting family of linear controllers as a single controller whose parameters are changed by monitoring the scheduling variables. Such a controller is called a gain-scheduled controller.

The concept of gain scheduling originated in connection with flight control systems. The nonlinear equations of motion of an airplane or a missile are linearized about selected operating points that capture the key modes of operation throughout the flight envelope. Linear controllers are designed to achieve the desired stability and performance requirements for the linearizations about the selected operating points. The parameters of the controllers are then interpolated as functions of gain scheduling variables; typical variables are dynamic pressure, Mach number, altitude, and angle of attack. Finally, the gain-scheduled controller is implemented on the nonlinear system.

5. Nonlinear Geometric Methods

A turning point in nonlinear control came in the 1980s with the development of the nonlinear geometric approach. Differential geometry proved to be an effective method for the analysis and design of nonlinear control systems. Synthesis problems like disturbance decoupling, non-interacting control, and output regulation have been dealt with using the tools of differential geometry. The most important achievements of the differential geometric approach have been in the study of controllability, observability and feedback linearization. In the 1960s, Kalman showed the important role of played by controllability and observability in linear control systems. Starting in the early 1970s research has been directed towards studying these concepts for nonlinear systems. The chapter on *Controllability and Observability of Nonlinear Systems* surveys the main approaches for studying controllability and observability of nonlinear systems, with emphasis on the differential geometric approach. The idea of feedback linearization appeared toward the end of the 1970s, motivated by physical examples like the computed torque method of robotic manipulators. The basic question of feedback linearization is: Can we transform a nonlinear system into an equivalent linear system by state feedback and/or a change of variables? The answer to this question takes the form of a set of simultaneous linear partial differential equations. The differential geometric approach made it possible to characterize the existence of a solution for these equations. More importantly, it led to the development of the concepts of relative degree, zero dynamics, and normal form, which permeate our current thinking about nonlinear systems. We will come back to these concepts in the next section. The main elements of the theory of feedback linearization are reviewed in the chapter on

Feedback Linearization of Nonlinear Systems.

Understanding the differential geometric approach requires knowledge of its tools; in particular, manifolds, Lie derivatives, Lie brackets, distributions, and the Frobenius theorem (see *Lie Bracket*). The chapter on *Differential Geometric Approach and Application of Computer Algebra* describes how computer algebra can be applied to perform the tests that arise in the differential geometric approach.

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Biographical Sketch

Hassan K. Khalil received the B.S. and M.S. degrees from Cairo University, Cairo, Egypt, and the Ph.D. degree from the University of Illinois, Urbana-Champaign, in 1973, 1975, and 1978, respectively, all in Electrical Engineering.

Since 1978, he has been with Michigan State University, East Lansing, where he is currently University Distinguished Professor of Electrical and Computer Engineering. He has consulted for General Motors and Delco Products.

He has published several papers on singular perturbation methods, decentralized control, robustness, nonlinear control, and adaptive control. He is author of the book Nonlinear Systems (Macmillan, 1992; Prentice Hall, 1996 and 2002), coauthor, with P. Kokotovic and J. O'Reilly, of the book Singular Perturbation Methods in Control: Analysis and Design (Academic Press, 1986; SIAM 1999), and coeditor, with P. Kokotovic, of the book Singular Perturbation in Systems and Control (IEEE Press, 1986). He was the recipient of the 1983 Michigan State University Teacher Scholar Award, the 1989 George S. Axelby Outstanding Paper Award of the IEEE Transactionson Automatic Control, the 1994 Michigan State University Withrow Distinguished Scholar Award, the 1995 Michigan State University Distinguished Faculty Award, the 2000 American Automatic Control Council Ragazzini Education Award, and the 2002 IFAC Control Engineering Textbook Prize. He is an IEEE Fellow since 1989.

Dr. Khalil served as Associate Editor of IEEE Transactions on Automatic Control, 1984 - 1985; Registration Chairman of the IEEE-CDC Conference, 1984; Finance Chairman of the 1987 American Control Conference (ACC); Program Chairman of the 1988 ACC; General Chair of the 1994 ACC; Associate Editor of Automatica, 1992-1999; Action Editor of Neural Networks, 1998-1999; and Member of the IEEE-CSS Board of Governors, 1999-2002. Since 1999, he has been serving as Editor of Automatica for nonlinear systems and control.