SOLUTION OF ELECTROMAGNETISM THEORY PROBLEMS

V.V. Denisenko
Institute of Computational Modeling SB RAS, Krasnoyarsk, Russia

Keywords: Electric field, magnetic field, mathematical modeling, Maxwell equations, electrostatics, magnetostatics, electric conductivity, elliptical equation, hyperbolic system of equations, variational principle, energy, potential, vector potential, dielectric, magnet, conductor, capacity, inductance, finite element method.

Contents

Summary

Maxwell equations are the basis of the theory of electromagnetic fields. In the stationary case they split into independent problems for electric and magnetic fields. For the electric field in conductive media electrical conductivity problem is formulated, and in insulators the problem of electrostatics is formulated. Equations for stationary problems are of elliptic type, whereas in general the system of Maxwell equations is hyperbolic.

A number of analytic methods of solution are known for stationary two-dimensional problems. For homogeneous isotropic media the solutions of these problems are obtained by the methods of functions of complex variable. Conformal domain mappings are widely used.

All the three stationary problems have equivalent variational formulations, that is, instead of solving differential equations it is possible to minimize a certain functional of energy over functions from certain set. Due to choice of finite subset of functions one can obtain particular approximate solutions by Ritz method or, in more general cases, by Galerkin method. When using set of grid functions this approach is called finite element method. In engineering practice this approach to a great extent replaced the difference methods, which are designed on the basis of approximation of differential equations. For this reason a great attention in the paper is paid to variational formulations of problems. Along with classical principles, starting from Dirichlet

principle, we state recent minimum principle of a quadratic energy functional for the case of asymmetric conductivity tensor, as well as undeservedly rarely used variational principles for nonlinear problems.

The relationship of the considered problems with the theory of electric circuits is shown. For the periodically varying fields, the problems of resonators and radio-wave reflection are formulated. For nonstationary Maxwell equations the general principles of formulation of initial boundary value problems are given.

A great variety of methods of mathematical modeling of electromagnetic phenomena specific for diverse branches of natural and technical sciences is emphasized.

1. Introduction

Electric and magnetic forces are present practically in all natural phenomena and determine the operation of many technical devices.

We consider here only the most general methods of mathematical modeling of electric and magnetic fields.

Numerous experiments show that the force acting an arbitrary body of small dimensions at an arbitrary point of space can be expressed as

$$\vec{F} = q\left(\vec{E} + \left[\vec{v} \times \vec{B}\right]\right) + m\vec{g} \ .$$

Here, it is implied that the body does not contact with other bodies and presents a macroscopic object, since at the scale of atomic nucleus the forces are of different type. The parameters q and m characterize the body, they are called charge and mass, repsectively. The vector $\vec{v}$ is the velocity of the body. The vector $\vec{g}$ is called gravity. It is mentioned only to emphasize the absence of other fundamental forces for macroscopic objects; we do not consider gravitation in the presentation that follows.

Vectors $\vec{E}$ and $\vec{B}$ are called electric field and magnetic induction, respectively. They are vector functions of space coordinates and time and describe certain physical object – electromagnetic field. The notion of electromagnetic field excludes direct interaction between charges. Charges create field in the whole space, and it acts on a charge at a particular point of space. Within the framework of special relativity, $\vec{E}$ and $\vec{B}$ are considered as components of unified tensor of electromagnetic field. It is convenient to leave $\vec{E}$ and $\vec{B}$ here, since the motion of particles is not considered.

According to the modern concepts, all the properties of electromagnetic field which are known from experiment can be deduced from the fact that vector functions $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ satisfy Maxwell equations:

$$\frac{\partial \vec{B}}{\partial t} + \text{rot}\,\vec{E} = 0$$
$$\varepsilon_0 \frac{\partial \vec{E}}{\partial t} - \frac{1}{\mu_0}\text{rot}\,\vec{B} = -\vec{J} \qquad (1)$$
$$\text{div}\,\vec{E} = \rho$$
$$\text{div}\,\vec{B} = 0.$$

Here $\vec{J}$ is current density vector, ρ is density of charge. In International System of Units constants are $\varepsilon_0 = 8.85 \cdot 10^{-12}\,\text{Fm}^{-1}$, $\mu_0 = 4\pi \cdot 10^{-7}\,\text{Hm}^{-1}$.

Applying operation div to the second equation and adding derivative with respect to time of the third equation, and taking account of $\text{div}\,\text{rot} \equiv 0$ one obtains restriction to right-hand sides, which is necessary for solvability of problem

$$\frac{\partial \rho}{\partial t} + \text{div}\,\vec{J} = 0. \qquad (2)$$

This is charge conservation law.

In conductors the current density is determined by Ohm’s law

$$\vec{J} = \sigma \vec{E}, \qquad (3)$$

where σ is conductivity. In isotropic media σ is scalar. In anisotropic media σ is symmetric tensor of second rank. A given function $\vec{J}_0$ can be added to right-hand side of (3). It describes density of currents of other nature. If characteristics of the medium are not invariant with respect to reflection, σ is asymmetric tensor. Example of this is semiconductor placed in strong magnetic field.

The symmetric part of σ always is positive definite tensor, since the density of heat energy production, which accompanies propagation of electric current through medium, is $\vec{E}^{\mathrm{T}} \sigma \vec{E}$. Here T denotes transposition.

In moving media Ohm’s law has the form

$$\vec{J} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right),$$

because electric field is transformed in this way when going over to coordinate system moving with the velocity of conductor $\vec{v}$.

Completion of Maxwell equations with motion equations of continuum gives equations of magnetohydrodynamics, which are often used for modeling of processes in liquid conductors. Use of kinetic equation for description of medium motion leads to Vlasov equations, which represent basic model in plasma physics.

Consideration of such models goes beyond the subject of the paper.

We suppose that right-hand sides of (1) and properties of the medium, in which electromagnetic field is considered, are given functions of coordinates and time, satisfying charge conservation law (2), or we use Ohm's law (3) to close the system of equations when considering conductive media.

When analyzing electromagnetic processes it is accepted to introduce auxiliary quantities, displacement vector

$$\vec{D} = \varepsilon \vec{E}, \tag{4}$$

where ε is permittivity, and magnetic intensity $\vec{H}$,

$$\vec{B} = \mu \vec{H}, \tag{5}$$

where μ is magnetic permeability of medium. For vacuum $\varepsilon = \varepsilon_0, \mu = \mu_0$.

Displacement $\vec{D}$ makes sense only in media with zero conductivity. Then current density $\vec{J}$ is not determined by Ohm's law (3), but is either a given function or determined by magnetization of the medium.

For anisotropic media ε and μ are not scalars, but symmetric tensors of the second rank.

Relationships (3), (4) and (5) are determined by the properties of the medium, and therefore they are called physical or material equations.

Tensors ε, μ always are positive definite, because they determine the energy density of the electromagnetic field

$$u = \frac{1}{2}\left(\vec{H}^T \vec{B} + \vec{E}^T \vec{D}\right). \tag{6}$$

Material equations (3), (4), (5) can be nonlinear as well. Nonlinear dependence $B(H)$ is typical for ferromagnets.

In nonlinear case the following relation between variations of energy density and variations of fields takes place:

$$\delta u = \vec{E}^T \delta \vec{D} + \vec{H}^T \delta \vec{B}. \tag{7}$$

Under constant ε, μ integration gives (6). Under linear dependences (4, 5) in some media additional terms $\vec{D}_0, \vec{B}_0$ in right-hand sides can be present. They describe nonvanishing intrinsic polarization and magnetization of substance under zero external fields. Permanent magnet is an example. In ferromagnets the dependence $\vec{H}\left(\vec{B}\right)$ is not only nonlinear, but ambiguous as well, since state of ferromagnet substance is determined by hysteresis.

Introduction of $\vec{D}$ and $\vec{H}$ enables us to exclude from explicit consideration, the redistributions of charges in dielectric and currents in magnetic. Maxwell equations (1) take the form:

$$\begin{aligned} &\frac{\partial \vec{B}}{\partial t} + \operatorname{rot} \vec{E} = 0 \\ &\frac{\partial \vec{D}}{\partial t} - \operatorname{rot} \vec{H} = -\vec{J}' \\ &\operatorname{div} \vec{D} = \rho' \\ &\operatorname{div} \vec{B} = 0, \end{aligned} \tag{8}$$

where right-hand sides ρ' and $\vec{J}'$, as distinct from (1), do not contain charge densities, appearing in dielectric, and current densities, appearing in magnetic, since polarization and magnetization are taken into account by material equations (4) and (5). Functions ρ' and $\vec{J}'$ also must satisfy charge conservation law (2).

Let turn to considerably simple and most interesting for applications statements of problems for systems of equations (8), (4), and (5) and (8), (3), and (5).

In stationary case, when all functions do not depend on time, system of equations (8), (4), and (5) splits into equations of electrostatics:

$$\begin{aligned} &\operatorname{rot} \vec{E} = 0 \\ &\operatorname{div} \vec{D} = \rho' \\ &\vec{D} = \varepsilon \vec{E} \end{aligned} \tag{9}$$

and magnetostatics:

$$\begin{aligned} &\operatorname{rot} \vec{H} = \vec{J}' \\ &\operatorname{div} \vec{B} = 0 \\ &\vec{B} = \mu \vec{H}. \end{aligned} \tag{10}$$

In conductive media instead of electrostatics (9) from (8), (3), and (5) equations of electrical conductivity are obtained

$$\operatorname{rot} \vec{E} = \vec{G}$$

$$\operatorname{div}\vec{J}=Q \qquad (11)$$
$$\vec{J}=\sigma\vec{E}.$$

The given functions on right-hand sides of (11) are usually equal to zero, but appear in some models to describe given magnetic field $\vec{B}_0$ which depends on time, $\vec{G}=-\frac{\partial\vec{B}_0}{\partial t}$, or currents of other nature with density $\vec{J}_0$, $Q=-\operatorname{div}\vec{J}_0$. Since $\operatorname{div}\operatorname{rot}\equiv 0$, for solvability of (11) it is necessary that

$$\operatorname{div}\vec{G}=0.$$

This requirement is provided due to $\operatorname{div}\vec{B}_0=0$.

In similar way, right-hand side $\vec{G}$ can be added to the first equation (9) too.

The system of equations (11) under $\vec{G}=0$ differs from (9) only by notations, and under $Q=0$ it likewise differs from (10), because coefficients ε,μ,σ, generally speaking, possess identical properties. In particular, for isotropic media all the three coefficients are positive scalar functions of coordinates.

If both right-hand sides $\vec{G},Q$ are different from zero, solution of problem (11) can be represented as sum of solutions of (9) and (10), with functions $\vec{G}$ and Q in place of $\vec{J}'$ and ρ'.

In practical problems nonlinearity is typical for μ mainly, and asymmetry – for σ. Therefore, let us consider the systems of equations of electrostatics, magnetostatics and electrical conductivity as independent, gradually complicating problems by introducing characteristic peculiarities.

In final sections of the paper we turn to more complex problems of fields harmonically depending on time, and fields, nonstationary in the true sense of the word.

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Biographical Sketch

V.V. Denisenko was born in Kiev, Ukraine. He completed his Diploma in Physics and Applied Mathematics at Novosibirsk State University, Novosibirsk, Russia in 1972. In 1979 he obtained the Candidate of Sciences degree at the Institute of Physics of the Russian Academy of Sciences Siberian Branch, Krasnoyarsk. In 1997 V.V. Denisenko was awarded the Doctor in Physics and Mathematics degree from Computing Center RAS SB, Novosibirsk with the thesis "Mathematical simulation of global distributions of electric field and current in the Earth's ionosphere". In 1972-1975 he was a postgraduate at the Computing Center RAS SB, Novosibirsk. Since 1975 he works at the Computing Center RAS SB, Krasnoyarsk as researcher, senior researcher and the head of the Laboratory of Mathematical Models for Near-Space. In 1997 the institute was renamed as the Institute of Computational Modeling RAS SB.
Dr V.V. Denisenko's research activities are the following ones: Mathematical simulation of the magnetospheric generators of electric field with electric energy dissipation in the ionosphere, Mathematical simulations of electric fields and currents in the Earth's ionosphere and magnetosphere with stresses on low latitude phenomena and magnetic variations. Creation of variational and numerical methods for nonsymmetrical boundary value problems of elliptical type which simulate transfer processes in gyrotropic or moving medium, Upgrading of multigrid methods for systems of partial differential equations of elliptical type.