NTU-GAMES

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Summary

Nontransferable utility (NTU) games derive from many economic situations. A classical example is an exchange economy. By pooling and redistributing their initial endowments, coalitions can reach certain payoff (utility) distributions that constitute the feasible set for that coalition. More generally, an NTU-game describes some feasible set of payoff vectors for every coalition. A solution predicts or prescribes a final payoff vector or a set of payoff vectors. Several solution concepts are reviewed in this survey. First, the strategically flavored concepts of the core and the bargaining set are described, and some existence results mentioned. An axiomatic characterization of the core based on a reduced game property, as well as a noncooperative approach to the core, are presented. The bargaining set is related to coalition structures and sheds some light on endogenous coalition formation. Second, several values for NTU-games are reviewed, starting with bargaining solutions and TU-values for the special cases of pure bargaining problems and TU (transferable utility) games. Characterizations are given of the Nash, Kalai-Smorodinsky and egalitarian bargaining solutions. The Shapley TU-value is characterized with the aid of dividends. Based on these ideas the Shapley solution, Harsanyi solution, Kalai-Samet solutions, Compromise solution, and Hart-Mas-Colell consistent solution are defined. Axiomatic characterizations of the Shapley, Harsanyi, and egalitarian NTU-solutions are presented as well.

1. Introduction

Game theory is concerned with models of conflict and cooperation between at least two parties. Cooperative game theory assumes that binding agreements between the players are possible, in contrast to noncooperative theory, where this is not the case. Because this is an implicit assumption, it is often more instructive to distinguish between the two approaches by considering the respective modeling techniques. In a noncooperative game the available information and moves of the players are described in detail. Given these moves, the possible strategies can be determined, and then, the payoffs are calculated resulting from various strategy combinations. The analysis of such a game is usually based on an equilibrium concept, notably Nash equilibrium. A cooperative game consists of a description of the payoffs that various coalitions can achieve, independently of the outside players. The latter presumes some assumption on what outside players will do. If, for instance, the players are firms in an industrial branch, then they may form a coalition (cartel). The total profit that the coalition can make depends on the behavior of the firms not belonging to the coalition. One possibility is that the coalition and its complement play a Nash equilibrium in a two-player noncooperative game with prices as strategies; but there are many others.

Cooperative games were first formulated by von Neumann and Morgenstern in their famous 1944 book, in order to deal with multi-person, constant sum, noncooperative games. Because the nice and clear-cut theory for two-person constant sum games, already developed by von Neumann in 1928, breaks down for more than two players, they computed for each coalition of players the maximin value that this coalition can obtain in the two-person constant sum game against its complement. In this way, the possibilities for each coalition are described simply by assigning to it a real number. More generally, a transferable utility (TU) game for n players is given by an array of real numbers, one for each of the 2^n possible coalitions (subsets) of players.

The expression 'transferable utility' refers to the assumption that there is some medium of exchange between the players, for instance money, and that the players' utilities are linear in money. The first condition is sometimes expressed as sidepayments being allowed. To avoid any confusion one can think about the numbers in a TU-game as expressing money. In a nontransferable utility (NTU) game, such a medium of exchange is not present or, if it is, the players' utilities are not linear in it. In an NTU-game, the possibilities for each coalition are represented by a set of utility ('payoff') vectors indexed by the members of the coalition. The implicit history of NTU-games goes back much further than 1944 because of their intimate relation (see below) with classical models of economies. As explicit objects, NTU-games were extensively studied starting from the sixties. The bulk of the literature, however, is devoted to the special cases of TU-games (as described above); and of pure bargaining games, which are NTU-games in which only the singleton coalitions (the individual players) and the grand coalition (of all players) play a role. The literature on NTU-games properly is much smaller, in particular if we abstract, as we do in this survey, from underlying economic models. The discussion on NTU-games as separate objects is somewhat complicated by the fact that there is no standard definition, whereas many results are sensitive to details in the definition of a game.

In this survey, we stick as much as possible to one definition of an NTU-game. Modifications are added or mentioned when needed or appropriate. The emphasis of the survey will be on the core and the bargaining set on the one hand, and on values for NTU-games on the other. The special cases of pure bargaining games and of TU-games are discussed separately: the main values for general NTU-games are extensions of the main solutions for these special case.

Notations Some notations that are used throughout are the following. The set of real (nonnegative real, positive real) numbers is denoted by $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$). For $x, y \in \mathbb{R}^N$ (where N is a finite set of natural numbers), $x > y$ means $x_i > y_i$ for every $i \in N$, and $x \geq y$ means $x_i \geq y_i$ for every $i \in N$. Also for $x, y \in \mathbb{R}^N$, $x \cdot y = \sum_{i\in N} x_i y_i$ denotes the inner product. For a subset X of $\mathbb{R}^N$, ∂X denotes its (topological) boundary. For sets A, B, $A \subseteq B$ denotes inclusion (possibly equality) and $A \subset B$ denotes strict inclusion (hence in particular $A \neq B$). Similarly for $A \supseteq B$ and $A \supset B$.

2. Basic Model and Definitions

A *game with nontransferable utility* or NTU-game is a pair (N, V) where $N \subset \mathbb{N}$ is a finite set, the set of *players*, and, for every *coalition* $S \subseteq N$, $V(S)$ is a subset of $\mathbb{R}^N$ satisfying the following four conditions:

(N1) If $S \neq \emptyset$, then $V(S)$ is non-empty and closed; and $V(\emptyset) = \emptyset$.
(N2) For every $i \in N$ there is a $V_i \in \mathbb{R}$ such that for all $x \in \mathbb{R}^N : x \in V(\{i\})$ if and only if $x_i \leq V_i$.
(N3) If $x \in V(S)$ and $y \in \mathbb{R}^N$ with $y_i \leq x_i$ for all $i \in S$ then $y \in V(S)$.
(N4) $\{x \in V(N) : x_i \geq V_i\}$ is a compact set.

The interpretation of such an NTU-game (N, V) is that $V(S)$ is the set of feasible payoff (utility) vectors for the coalition S if that coalition forms. Only the coordinates for players $i \in S$ in elements of $V(S)$ matter. The embedding in $\mathbb{R}^N$ is for convenience and by (N3), it is 'cylindrical'. Another consequence of (N3) is that, if x is feasible for S then also any $y \leq x$ is feasible for S; this property is often called *comprehensiveness*, and it can be interpreted as free disposibility of utility. Condition (N4) ensures that the *individually rational* part of $V(N)$ is bounded (it is closed by (N1)).

The collection of all NTU-games with player set N (satisfying (N1)-(N4)) is denoted by $\mathcal{G}^{\mathcal{N}}$. Sometimes, if confusion is not likely, V will be written instead of (N, V).

Examples of NTU-games are the following.

Example 1 (*Market Game*)

Each player $i \in N$ is an economic agent, endowed with a bundle $e_i \in \mathbb{R}^k_+$ of k goods. Agent i's preferences are described by a continuous utility function $u_i : \mathbb{R}^k_+ \to \mathbb{R}$. Each coalition S can pool its resources to a bundle $\sum_{i \in S} e_i$ and exchange (redistribute) this bundle between its members. This situation gives rise to an NTU-game (N, V) with $V(\varnothing) = \varnothing$ and for every nonempty coalition S:

$$V(S) := \Big\{ x \in \mathbb{R}^N : \text{there exist } y_i \in \mathbb{R}^k_+,\ i \in S, \\ \text{with } \textstyle\sum_{i \in S} y_i \leq \sum_{i \in S} e_i \text{ and } x_i \leq u_i(y_i) \text{ for all } i \in S \Big\}. \tag{1}$$

It is easy to check that (N, V) is indeed an NTU-game. It is usually called a *market game*.

Example 2 (*Pure Bargaining Game*)

Suppose there is one unit of a perfectly divisible good to distribute between the agents in N. These agents are characterized by utility functions $u_i : [0, 1] \to \mathbb{R}$ $(i \in N)$. If the players reach an unanimous agreement on a division of the good then the corresponding n-tuple of utilities is the outcome; otherwise the game ends in disagreement, meaning that each player i receives nothing, i.e., ends up with utility $u_i(0)$. This results in an NTU-game (N, V) with

$$V(N) = \{ x \in \mathbb{R}^N : \exists \alpha_i \geq 0 (i \in N), \sum_{i \in N} \alpha_i = 1 [x_i \leq u_i(\alpha_i) \forall_i \in N] \} \tag{2}$$

and for $S \neq N$:

$$V(S) = \left\{ x \in \mathbb{R}^N : \forall i \in S \left[x_i \leq u_i(0) \right] \right\}. \tag{3}$$

This means that only the grand coalition of all players can gain from cooperation. Usually, such a game is described by a pair (B, d) where $B = V(N)$ is the *feasible set* and $d \in B$ is the *disagreement outcome*, i.c., $d = (u_i(0))_{i\in N}$.

Example 3 (*Games with Transferable Utility*)

Suppose that in Example 1 the agents' utilities are expressed in some common medium, say money, and that, moreover, each agent's utility for money is equal simply to the amount that player has. In that case, it would be natural to define

$$V(S) := \left\{ x \in \mathbb{R}^N : \sum_{i\in S} x_i \le \max\left\{ \sum_{i\in S} u_i(y_i) : \sum_{i\in S} y_i \le \sum_{i\in S} e_i \right\}\right\}. \tag{4}$$

It is usual to denote the maximum of the sums of utilities in this definition by $v(S)$. More generally, a pair (N, v) where $v : 2^N \to \mathbb{R}$ with $v(\varnothing) := 0$ is called a game with transferable utility or TU-game. Such a game may arise from many situations, not only from exchange economies.

In this survey, the emphasis will be on 'abstract' NTU-games. The question which NTU-games are market games, i.e., for which NTU-games we can find an underlying exchange economy giving rise to these NTU-games, was considered by Billera and Bixby in the early 1970s.

3. The Core of an NTU-Game

The core of an NTU-game (N, V) consists of all payoff vectors that are feasible for the grand coalition N and that cannot be improved upon by any coalition, including N itself. If $x \in V(N)$, then S can *improve upon* x if there is a $y \in V(S)$ with $y_i > x_i$ for all $i \in S$. Hence the formal definition of the core of the game (N, V) is

$$C(N,V) = V(N) \setminus \bigcup_{S\subseteq N} \operatorname{int} V(S), \tag{5}$$

where 'int' denotes the topological interior.

We first consider some examples.

Example 4

In a pure bargaining game (B, d) (see Example 2) the core consists of all boundary points x of $B = V(N)$ with $x \ge d$. For such games the core is not a very restrictive concept. In a game with transferable utility (N, v) the core can be written as

$$C(N,v) = \left\{ x \in \mathbb{R}^N : \sum_{i\in N} x_i = v(N), \qquad \forall S \subseteq N \left[\sum_{i\in S} x_i \ge v(S) \right]\right\}. \tag{6}$$

Example 5

Consider an exchange economy as in Example 1 with two goods (amounts denoted by y^1 and y^2) and three agents ($N = \{1, 2, 3\}$) and assume that the utility functions are given by $u_i(y^1, y^2) = \min\{a_i y^1, y^2\}$ for every $i = 1,2,3$, where $a_1 = a_2 = 1$ and $a_3 = \frac{1}{2}$.

Let the endowments be $e_1 = (10, 0)$, $e_2 = (5, 5)$, and $e_3 = (0, 10)$. The corresponding NTU-game is described by

$$V(\{i\}) = \{x \in \mathbb{R}^N : x_i \le 0\} \quad \text{for } i = 1, 3 \tag{7}$$

$$V(\{2\}) = \{x \in \mathbb{R}^N : x_i \le 5\} \tag{8}$$

$$V(\{1, 2\}) = \{x \in \mathbb{R}^N : x_1 + x_2 \le 5,\ \ x_1 \le 5,\ \ x_2 \le 5\} \tag{9}$$

$$V(\{1, 3\}) = \{x \in \mathbb{R}^N : x_1 + 2x_3 \le 10,\ \ x_1 \le 10,\ \ x_3 \le 5\} \tag{10}$$

$$V(\{2, 3\}) = \left\{x \in \mathbb{R}^N : x_2 + 2x_3 \le 5,\ \ x_2 \le 5,\ \ x_3 \le \frac{5}{2}\right\} \tag{11}$$

$$V(N) = \left\{x \in \mathbb{R}^N : x_1 + x_2 + 2x_3 \le 15, x_1 \le 15, x_2 \le 15, x_3 \le \frac{15}{2}\right\}. \tag{12}$$

The core of this game is the set $C(N, V) = \{x \in \mathbb{R}^N : x \ge 0, x_2 = 5, x_1 + 2x_3 = 10\}$.

Example 6

Consider the three-person NTU-game defined by $N = \{1, 2, 3\}$ and

$$V(N) = \{x \in \mathbb{R}^N : x_1 \le 0.5, x_2 \le 0.5, x_3 \le 0\}, \tag{13}$$

$$V(\{1, 2\}) = \{x \in \mathbb{R}^N : x_1 + x_2 \le 1\}, \tag{14}$$

and

$$V(S) = \{x \in \mathbb{R}^N : x_i \le 0, \text{for all } i \in S\} \tag{15}$$

otherwise. The core of this game equals $C(N, V) = \{(0.5, 0.5, 0)\}$.

3.1 Existence of the Core

One of the main questions concerning the core of an NTU-game is its existence. For a pure bargaining game that question is easy to answer, see Example 4. For an NTU-game, the concept of balancedness plays an important role.

Start with a TU-game (N, v) and assume that $x \in C(N, v)$. Let $S_1, ..., S_k$ be a partition of the player set N, then it follows immediately from the definition of the core that

$$v(N) = \sum_{i \in N} x_i = \sum_{j=1}^{k} \sum_{i \in S_j} x_i \geq \sum_{j=1}^{k} v(S_j), \tag{16}$$

hence, $v(N) \geq \sum_{j=1}^{k} v(S_j)$ is a necessary condition for nonemptiness of the core. Similarly, by summing other core constraints for x, one shows that for instance

$$v(N) \geq \sum_{i \in N} \frac{1}{n-1} v(N \setminus \{i\}) \tag{17}$$

must hold as well. More generally, we call a collection of coalitions $\mathcal{S} \subseteq 2^N \setminus \{\emptyset\}$ *balanced* if there are positive numbers λ_S for $S \in \mathcal{S}$ such that $\sum_{S \in \mathcal{S}: S \ni i} \lambda_S = 1$ for every $i \in N$. These numbers are called *balancing weights*. One can think of a balanced collection as of a generalized partition.

The given instances indicate that for a TU-game (N, v) the condition

$$v(N) \geq \sum_{S \in \mathcal{S}} \lambda_S v(S) \tag{18}$$

for every balanced collection $\mathcal{S}$ with balancing weights λ_S, $S \in \mathcal{S}$, is necessary for (N, v) to have a non-empty core. Bondareva and Shapley proved, using the duality theorem of linear programming, that this condition is also sufficient. A TU-game satisfying (18) is called *balanced*, hence: A TU-game has a nonempty core if and only if it is balanced.

More generally, an NTU-game (N, V) is called *balanced* if the condition

$$\bigcap_{S \in \mathcal{S}} V(S) \subseteq V(N) \tag{19}$$

holds for every balanced collection $\mathcal{S}$. It is not hard to check that (18) and (19) are equivalent for a TU-game, viewed as a special NTU-game. For an NTU-game, however, balancedness is no longer a necessary condition for nonemptiness of the core. In Example 6, for instance, $(1, 0, 0) \in V(\{1, 2\}) \cap V(\{3\})$ but $(1, 0, 0) \notin V(N)$; since $\{\{1,2\},\{3\}\}$ is a balanced collection, the game is not balanced. Nevertheless, it has a non-empty core. Balancedness is, however, still a sufficient condition:

Theorem 1

Every balanced NTU-game has a nonempty core.

The NTU-game in Example 5 is an example of a balanced game, as can be checked. More generally, consider the market game (N, V) in Example 1 and assume that the underlying preferences of the agents are convex, i.e., the utility functions u_i are quasiconcave (as is the case in Example 5). Let $\mathcal{S}$ be a balanced collection and $x \in \bigcap_{S\in\mathcal{S}} V(S)$. This means that there are bundles of goods y_i^S for each $i \in S$ and $S \in \mathcal{S}$ with $\sum_{i\in S} y_i^S \le \sum_{i\in S} e_i$ and with $x_i \le u_i(y_i^S)$ for every $S \in \mathcal{S}$ and $i \in S$. Let λ_S be balancing weights for $\mathcal{S}$ and define, for every $i \in N$, $y_i^N := \sum_{S\in\mathcal{S}:i\in S} \lambda_s y_i^S$. Then

$$\sum_{i\in N} y_i^N = \sum_{i\in N}\sum_{S\in\mathcal{S}:i\in S} \lambda_s y_i^S \le \sum_{i\in N}\sum_{S\in\mathcal{S}:i\in S} \lambda_s e_i = \sum_{i\in N} e_i\ , \tag{20}$$

where the last equality follows since $\mathcal{S}$ is a balanced collection. Thus, y^N is a feasible redistribution of the total endowments of the grand coalition. Further, for every $i \in N$, by quasiconcavity of u_i,

$$x_i \le \min_{s\in\mathcal{S}:i\in S} u_i(y_i^S) \le u_i\left(\sum_{s\in\mathcal{S}:i\in S} \lambda_s y_i^S\right) = u_i(y_i^N), \tag{21}$$

so that $x \in V(N)$. This shows that (N, V) is balanced. By Theorem 1 we have:

Theorem 2

Every market game with convex preferences has a nonempty core.

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Biographical Sketch

Hans Peters is Professor of Mathematical Economics at the Department of Quantitative Economics of the University of Maastricht, The Netherlands. He received his master's and doctoral degrees in Mathematics from the University of Nijmegen. He has done extensive research in the areas of Game Theory and Social Choice Theory with emphasis on bargaining, cooperative game theory, and strategy-proofness issues in social decision making. Professor Peters has more than sixty publications, including articles in scientific journals such as *Econometrica, Journal of Economic Theory, Mathematics of Operations Research, Operations Research, Management Science, Journal of Mathematical Economics, International Journal of Game Theory, Games and Economic Behavior, Social Choice and Welfare, Mathematical Social Sciences, Discrete Mathematics, Journal of Economic Behavior and Organization, Economics Letters,* and *Theory and Decision.* He is the author of the book *Axiomatic Bargaining Game Theory,* published by Kluwer in the *Theory and Decision Library Series C,* of which he recently became the main editor. Professor Peters is associate editor of the *Games and Economic Behavior, Social Choice and Welfare,* and *Mathematical Social Sciences.* He is a member of many scientific communities, among which are the *Society for Social Choice and Welfare* and the *Game Theory Society.*