DECISION PROBLEMS AND DECISION MODELS

J. Sounderpandian
Department of Business, University of Wisconsin – Parkside, USA

Keywords: Decision, decision criteria, risk, uncertainty, ambiguity, expected utility, non-expected utility, rank-dependent expected utility, voting theory, decision tree, influence diagram.

Contents

Summary

A decision is defined as choosing from a set of available alternatives the best one that optimizes a set of decision criteria. Decision problems can be quite varied and quite complex. An important first step in the solution of a problem is to find an applicable solution model. Fortunately, powerful solution models are available to solve most, if not all, classes of decision problems. A knowledge of the different classes of problems and the variety of techniques that can be used to solve each class of problems will therefore be useful. The aim of this article is to provide that knowledge.

A decision problem can be classified based on the number of decision makers, the cardinality of available alternatives, the decision criteria, the nature of environment in which the decision is to be made, the nature of uncertainty involved and whether it is a single or a series of decisions. The classification helps in identifying possible difficulties and pitfalls in solving a given class of problems. It also helps to choose a suitable solution technique.

Axiomatic descriptions of two important decision theories, namely, the Expected Utility theory and the Rank-Dependent Expected Utility are provided. Following that a very short description of the essentials of Voting Theory is given. The utility theories call for the use of a few different solution techniques. In that connection, a short description of Decision Trees and Influence Diagrams is given. Some hints about structuring the problem, and aggregating probability judgments are also included.

There are a few limitations of decision models. For example, there is no model that can tell us what is the correct level of risk aversion in a given situation for a given decision maker. Nor can a model tell us how to “surprise” a competitor.

1. Introduction

A decision problem is one in which we have to find the best alternative from a set of available alternatives. This statement immediately leads to the question, what exactly is the ‘best alternative.’ One way to answer this question is to assume the existence of a binary preference relation $\succcurlyeq$ on the set of alternatives, *A*. For $a, b \in A$, $a \succcurlyeq b$ denotes that the decision maker prefers *a* to *b* or is indifferent between them; $a \succ b$ denotes that *a* is strictly preferred to *b*; and $a \approx b$ means that they are equally preferred. The best alternative then is the one that is most preferred. That is, if *b* is the best alternative then there is no other alternative in *A* that is strictly preferred to *b*. The best alternative exists when *A* is finite and $\succcurlyeq$ is a weak order (complete, reflexive and transitive). The problem is simple if *A* is finite and $\succcurlyeq$ is a weak order, because it then boils down to a simple search. In a typical decision problem $\succcurlyeq$ on *A* is not completely known and therefore a solution model is needed. We will assume $\succcurlyeq$ is a weak order, though we will have occasion to relax that assumption later.

Often a problem can be analyzed better by moving it into a *criteria space*. For example, the choice among available medicines for an illness can be analyzed in the 3-dimensional criteria space: effectiveness, cost and side effects. Suppose there is a criteria space *B*, such that every alternative *a* in *A* can be mapped to a point *m(a)* in *B* through a *measurement* function $m: A \to B$. The decision maker may have a preference relation $\succcurlyeq$ on *B* that is known completely and is easier to handle mathematically. But an important consistency condition has to hold before we can exploit $\succcurlyeq$ on *B*. To distinguish the two $\succcurlyeq$'s on *A* and *B* we shall use superscripts, such as $\succcurlyeq^{A}$. For consistency (or *order preservation*) between $\succcurlyeq^{A}$ and $\succcurlyeq^{B}$, we must have

$$\forall a_1, a_2 \in A : a_1 \succcurlyeq^{A} a_2 \Leftrightarrow m(a_1) \succcurlyeq^{B} m(a_2) \tag{1}$$

The criteria should be chosen carefully so that the consistency condition (1) is satisfied. Unfortunately, it is difficult to verify whether (1) holds by checking every pair (a_1, a_2), because the number of alternatives can be large or infinite. We then have no other course but to simply believe that (1) holds. Difference in beliefs can lead to lively debates. To illustrate, suppose *A* is a set of public investment alternatives and the criteria space *B* has just one dimension, namely, the net present value (NPV) of the investment. Later it appears that it is possible that two alternatives with the same NPV are not equally preferable because one alternative is more harmful to the environment than the other. We then have to expand *B* to include the environmental effects in order to satisfy (1). On the other hand, some people may believe that environmental effects are irrelevant, at least for the set *A* in consideration. This could lead to public debates about what the relevant criteria are. In such cases, the only solution seems to be to adopt what the public debate settles.

When (1) is satisfied, we shall write $\succcurlyeq^{A} \stackrel{m}{\rightarrow} \succcurlyeq^{B}$. A binary weak order that everyone is familiar with is the usual order ≥ on the set of real numbers, $\Re$. If we can have a *utility function u* such that $\succcurlyeq^{B} \stackrel{u}{\rightarrow} \geq^{\Re}$, then *u* makes the preferences easily understood and handled. But this again requires a consistency condition similar to (1), between $\succcurlyeq^{B}$ and $\geq^{\Re}$, to be satisfied. The condition is:

$$\forall b_1, b_2 \in B : b_1 \succcurlyeq^{B} b_2 \Leftrightarrow u(b_1) \geq^{\Re} u(b_2) \tag{2}$$

Once again verifying (2) can be impossible. But unlike the case of (1), (2) tends to be on a more formal ground, because *B* tends to be more formally defined than *A*. As a result, a proponent who wants to convince us that (2) holds could present us with an axiomatic proof of (2). Typically, the proponent will posit a set of axioms about $\succcurlyeq^{B}$, and give a formal proof that (2) follows from those axioms. All that this axiomatic proof does is, of course, transfer the belief from (2) to the axioms. But if the axioms are appealing as self-evident, we would have a good reason to believe in (2). In the theories and models presented later in this article, we will pay special attention to the underlying axioms.

A utility function, *u*, will be enormously useful in converting a decision problem into a mathematical programming problem. Note that $\succcurlyeq^{A} \stackrel{m}{\rightarrow} \succcurlyeq^{B} \stackrel{u}{\rightarrow} \geq^{\Re}$ implies $\succcurlyeq^{A} \stackrel{um}{\rightarrow} \geq^{\Re}$ where *um* denotes *u*(*m*()). The aim of a typical decision model is to reduce $\succcurlyeq^{A}$ to $\geq^{\Re}$; that is, attach a number, or utility, to each alternative so that a more preferred alternative will have a larger utility. We can then pick the best alternative by looking for the largest utility.

Decision problems can be quite varied, complex and daunting. Fortunately, some very great thinkers – economists, operations researchers, psychologists, mathematicians and philosophers – have developed impressive theories and models that can be applied to solve many types of problems. We shall first classify decision problems, often with reference to the alternatives space *A* and the criteria space *B*. We will then use this classification to match decision models to the different classes of problems.

2. A Classification of Decision Problems

2.1 The Number of Decision Makers

The first classification is based on how many decision makers there are. In general, the problem is simpler when there is only one decision maker than when there are many. One important reason for that is Arrow's Impossibility Theorem (AIT). As the name implies, AIT is a negative result. Consider a finite set A of alternatives containing k (> 2) alternatives and n (>1) decision makers. For each decision maker i, $1 \leq i \leq n$, let $\succcurlyeq_i$ denote that person's binary preference relation on A. That is, for $a_1, a_2 \in A$ we will write $a_1 \succcurlyeq_i a_2$ to denote that person i prefers a_1 to a_2 or is indifferent between them. AIT concerns how we aggregate all the $\succcurlyeq_i$ to get a group preference relation $\succcurlyeq$. Note that the only criteria used in this aggregation method are the individual rankings, $\succcurlyeq_i$, of the alternatives. Whether other criteria should be included is an issue we will soon consider.

We can describe AIT as follows. Consider the following rationality conditions on the aggregation process.

AIT 1. *Completeness*: For any input data consisting of A and all the $\succcurlyeq_i$ on A, the aggregation method must produce a $\succcurlyeq$ on A.

AIT 2. *Positive Association*: Suppose we have $a_1 \succcurlyeq a_2$. Now if some individual i who originally had $a_2 \succcurlyeq_i a_1$ changes her mind to $a_1 \succcurlyeq_i a_2$ then we must continue to have $a_1 \succcurlyeq a_2$.

AIT 3. *Independence of Irrelevant Alternatives*: Suppose we have $a_1 \succcurlyeq a_2$. If a third alternative a_3 is removed from A, and on $A \setminus \{a_3\}$ all the $\succcurlyeq_i$ remain the same as before, then in the new $\succcurlyeq$ on $A \setminus \{a_3\}$ we must have $a_1 \succcurlyeq a_2$.

AIT 4. *Sovereignty*: For any two alternatives a_1 and a_2, there must be some input data of individual rankings $\succcurlyeq_i$ on A for which we get $a_1 \succcurlyeq a_2$.

AIT 5. *Non-Dictatorship*: For any individual i, it should not be the case that $a_1 \succcurlyeq_i a_2 \Rightarrow a_1 \succcurlyeq a_2$ irrespective of how the others rank a_1 and a_2.

Taken individually, each of the above conditions is quite appealing as a rationality requirement and appears to be easily satisfiable. But, Kenneth Arrow proved that it is impossible to satisfy all of them in any aggregation method. Later, others showed that relaxing one of more of the five conditions does not lead to efficient aggregation methods.

Given these negative results, we naturally look for aggregation methods that use additional information, such as the *intensity of preferences*. For example, we could define for each individual i a utility function u_i: $A \rightarrow \Re$ such that $a_1 \succcurlyeq_i a_2 \Leftrightarrow u_i(a_1) \geq u_i(a_2)$, and similarly define a group utility function $u = u(u_1, u_2, \ldots, u_n)$ to reflect the aggregated preference. It has been shown that the additive group utility

$$u(a) = \sum_{i=1}^{n} k_i u_i(a) \tag{3}$$

is the only aggregation method that would

1. satisfy AIT 1 – AIT 5 that are re-written in terms of u and u_i, and
2. be applicable per the expected utility theory (described later).

While it is remarkable that a simple additive form is what would work for u, it is also troublesome that this u makes a very explicit form of *interpersonal comparisons of utility*. To wit, the k_i values used in (3) can be controversial because they determine the trade-off between individual utilities. There can easily be disagreement among the individuals about these k_i values.

Another issue in group decision making is whether the individuals are cooperative or not. Cooperation can eliminate certain fears of an individual about adverse actions by others, and thus will increase the alternatives available to the group. As a result, it can improve the overall utility of the best choice. The aggregation of preferences we discussed above obviously requires cooperation in that the individuals have to agree to the result of the aggregation. Non-cooperation can give rise to unpredictable competitive behavior of the individuals. Finding the best group decision, allowing for unpredictable competitive behavior, is difficult.

2.2 The Number and Nature of the Criteria

The number of dimensions of the criteria space B is relevant when it comes to finding a utility function u such that $\succsim^{B} \xrightarrow{u} \geq^{\Re}$. Also, each dimension can be discrete or continuous. The simplest case is when the number of dimensions is 1, as it avoids having to estimate tradeoffs between criteria. A common uni-dimensional criterion is money, because anything fungible can be reduced to money. An advantage of reducing the dimension(s) to money is that a whole slew of financial techniques can be brought to bear on the problem. When a criterion is not fungible, such as the health of the decision maker, we are on a different ground. Techniques such as portfolio formation, hedging through options & futures are not feasible or of only limited use. Furthermore, market forces such as price and interest rate are absent.

When $B = \Re^n$, u is called a *multi-criteria utility function*. The simplest such function is the *additive* u where for $x = (x_1, x_2, \ldots, x_n) \in B$,

$$u(x) = \sum_{i=1}^{n} k_i v_i(x_i) \tag{4}$$

where k_i is the weight given to the i^{th} criterion and all v_i: $\Re \rightarrow \Re$ are defined so that $v_i(x_i)$ is the utility contributed by x_i amount of the i^{th} criterion. An additive utility function is obviously preferable since it will simplify finding the optimal decision.

Can we always find a set of additive criteria for a given problem? Surprisingly, almost always such a set exists, but, alas, we may not be able to find it. A theorem due to Kolmogorov proves that if a criteria space *C*, a measurement function *m*' and a *continuous u*' exist such that $\succcurlyeq^{A} \overset{m'}{\rightarrow} \succcurlyeq^{C} \overset{u'}{\rightarrow} \geq^{\Re}$ then there exist a criteria space *B*, a measurement function *m*, and an *additive* and continuous utility function *u* such that $\succcurlyeq^{A} \overset{m}{\rightarrow} \succcurlyeq^{B} \overset{u}{\rightarrow} \geq^{\Re}$. Thus, the only prerequisite for additive criteria is the existence of some set of criteria with respective to which there exists a continuous utility function. Since continuity is a very weak condition, we can say that for almost all decision problems there exists a set of additive criteria. Although the proof of Kolmogorov's theorem is constructive, the construction is too mathematical to prescribe a practical method for constructing an additive set of criteria. But the theorem is encouraging, because the same theorem forms the basis for the impressive abilities of neural networks. Thus, it points to the possibility that we can think through a problem, use our experiences and come up with an additive set of criteria.

Although constructing a set of criteria can be difficult, given a set of criteria we can easily check to see if it is additive. A necessary condition for additivity is *preferential independence*. The condition can be described as follows. Let $I \subseteq \{1, 2, \ldots, n\}$ and let $\Delta^{I} \in \Re^{n}$ denote a vector such that $\Delta_i = 0$ for $i \notin I$ and $\Delta_i \neq 0$ for $i \in I$. Preferential independence requires

$$\forall x, y \in B, I \text{ and } \Delta^{I} : \left(x + \Delta^{I}\right) \succcurlyeq^{B} x \Rightarrow \left(y + \Delta^{I}\right) \succcurlyeq^{B} y \tag{5}$$

This is a strong condition, and often may not be satisfied. As a result, *u* will have to be non-additive. The class of non-additive functions is large and varied. A few examples can be given.

One possible case is interaction among the dimensions. Interactions can be taken care of by including interaction terms which are products of the different dimensions. For example, with two dimensions, we can have

$$u(x) = k_1\, u_1(x_1) + k_2\, u_2(x_2) + k_{12}\, u_1(x_1) u_2(x_2) \tag{6}$$

With more dimensions, we may have to have many more interactions terms.

The next is the *maximin* function where the problem is to maximize the minimum among a set of values. Rawls's *Difference Principle* which equates the welfare of a society to that of the worst-off individual in that society is a case in point. The utility function then is

$$u(x) = \text{Min}_i \left[u_i(x_i)\right] \tag{7}$$

An extension of the maximin function is to combine it with the additive case to get what is called the *min-sum* function. An example is

$$u(x) = \text{Min}\left[u_1(x_1), u_2(x_2)\right] + u_3(x_3) \tag{8}$$

The case (7) means that all the dimensions are *complements*, such as the links in a chain, implying a chain is only as strong as the weakest link. Case (8) means that dimensions 1 and 2 are complements, and dimension 3 is a *substitute* for dimensions 1 and 2 together. In (4), all dimensions are substitutes for one another. Complements and substitutes can combine in complex ways giving rise to a min-sum function that needs quite a few terms.

Our final example is the *lexicographic* case where there is a hierarchy among the dimensions, and with respect to that hierarchy the preference is lexicographic. For example in a rescue operation, the number of lives saved will be hierarchically superior to the amount of materials saved in that no amount of material saved can compare to a life saved. The preference then is lexicographic. A combination of lexicographic and maximin preferences yields the *leximin* preference. The goal in the leximin case is to maximize the minimum in the hierarchically first dimension and then that of the next dimension and then the next, and so on.

The utility function defined on the criteria space in one of the above forms can serve as the objective function in a mathematical programming problem that is designed to find the best alternative.

2.3 The Temporal Aspects of Decisions

Another aspect of the decision problem that is relevant to the solution process is the timing of decisions. If the decision is just one choice to be made at a given time, then that is the simplest case. If several related decisions have to be made serially over a period of time, in stages, then we have a more complex problem. One reason for the complexity is that the number of alternatives available increases exponentially with the number of stages. For example if a problem has s number of stages and at each stage n alternatives are available, we then have to examine n^s total alternatives.

Additional issues arise in serial decisions in the form of *dynamic consistency*. For example, the best *sequence* of decisions, given the decision maker's preferences at the start may be to make the sequence of choices $c_1, c_2, \ldots c_s$ at the s number of stages. But after c_1 is made and depending on events that happen before the next choice, it is quite possible that the decision maker's preferences change. And according to the new preferences $c_2, c_3, \ldots c_s$ may not be optimal sequence anymore. Suppose the decision maker keeps changing the remaining sequence at will at each stage, she can end up with a very bad final position. How do we solve this problem? A little thought reveals that if we treat the decision maker at different times as different individuals, then the problem resembles a group decision problem. That means we need to make controversial "interpersonal" comparisons to find the best sequence. Even after we manage to do that, it is not clear how the individual would stick to the best sequence. (Recall the legend about Ulysses and the sirens.) Since we have so much difficulty with just one individual, we are bound to have much more difficulties with a serial group decision problem. It will call for a lot of coordination among the group members.

2.4 The Nature of Uncertainty

Almost every decision involves uncertainties which may arise due to different types of reasons. Let us see some of them.

First the uncertainty could be due to our ignorance about the future. We cannot predict the outcome of a coin toss. Often we face a more complex future event such as the outcome of a sports event, the future path of a storm, tomorrow’s weather, next week’s demand for cars or next month’s interest rate. Sometimes the uncertainty exists because of lack of necessary information. For example, we may not know whether there is oil at a proposed drilling site. Some other times the uncertainty is due to competitive behavior. For example, a business manager implementing a marketing strategy may not know what her competitor might do in the meantime.

Among the above types of uncertainties, the outcome of a coin toss is the simplest to handle. We can assign a probability of 0.5 to each outcome if we believe the coin is fair; or we can toss the coin a sufficient number of times and assign the probabilities based on the frequencies observed. The hardest case is competitive behavior. A competitor we face mostly wants to surprise us; thus our probability assignment to his possible actions can easily go wrong. Only in some restricted circumstances, as in some forms of idealized games, will we be able to assign probabilities to a competitor’s actions.

In other intermediate cases, such as the path of a severe storm, it is still hard to assign probabilities for all possible outcomes. One way to do it is to aggregate the opinions of experts. Depending upon how important the decision is, certain minimum number of experts must be involved.

For important public decisions, it is common to use as many as 20 experts. Many different methods of aggregating expert opinions are available. The simplest and the most common method is to take the weighted average of the probabilities, with weights determined by the decision maker(s).

When probabilities cannot be assigned readily to the uncertain events, due to whatever reason, we say that there is *ambiguity*. It has been empirically shown that besides risk aversion, an average person also exhibits ambiguity aversion. It may therefore be necessary to allow for ambiguity aversion while prescribing a solution.

-
-
-

Bibliography

Bhatti, M. A. A. and Bhatti, M. A. (1999). *Practical Optimization Methods.* 715 pp. Springer-Verlag, New York, NY. [EU maximization can call for mathematical programming. This book is then a good reference.]

Fishburn P. (1970). *Utility Theory for Decision Making.* 234 pp. Wiley, New York, NY. [Excellent reference for *theoretical* aspects of EU theory.]

Keeney R. L. and Raiffa H. (1976). *Decisions with Multiple Objectives: Preferences and Value Trade-offs.* 569 pp. Wiley, New York, NY. [Standard reference for theoretical and practical aspects of EU theory and multicriteria decision theory.]

Myerson, R. B. (1997). *Game Theory: Analysis of Conflict.* 584 pp. Harvard University Press, Boston, MA. [Among the many books on the subject, this one has the best combination of readability, rigor and coverage.]

Nurmi H. (1999). *Voting Paradoxes and How to Deal with Them.* 153 pp. Springer-Verlag, New York, NY. [The title says it all.]

Quiggin J. (1993). *Generalized Expected Utility Theory: The Rank-Dependent Model*. 208 pp. Kluwer Academic Publishers, Boston, MA. [Contains an authoritative description of the RDEU theory.]

Raiffa H. (1968). *Decision Analysis: Introductory Lectures on Choices under Uncertainty*. 309 pp. Addison-Wesley, Reading, MA. [A reader-friendly book on theoretical and practical aspects of Decision Trees and EU theory.]

Sounderpandian J. (1992). Transforming Continuous Utility into Additive Utility Using Kolmogorov's Theorem. *Journal of Multi-Criteria Decision Analysis.* **1**, 93-99. [Proves the existence of additive criteria for multicriteria problems.]

Biographical Sketch

Dr. Jayavel Sounderpandian is a professor of Quantitative Methods at the University of Wisconsin-Parkside. He teaches Operations Management and Business Statistics. He received his bachelor's degree in mechanical engineering from Indian Institute of Technology, Madras, and his master's and doctoral degrees in business administration from Kent State University. His main area of research is Decision Analysis, and has published in *Operations Research*, *Journal of Risk and Uncertainty, Interfaces*, *Abacus*, *Journal of Multicriteria Decision Analysis* among several others. He has written two books on the use of spreadsheets in business applications. Currently he is co-authoring a textbook on business statistics.