

# STRUCTURAL ANALYSIS

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## Summary

This chapter presents an overview of the modern method of structural analysis based on discrete modeling methods. Discrete structural modeling is suited for digital computation and has led to the generalization of the formulation procedure. The two principal methods, namely the matrix force method and the matrix displacement method, are convenient for analysis of frames made up mainly of one-dimensional members. Of the two methods, the matrix displacement method is more popular, due to its natural extension to the more generalized finite element method. Using displacements as the primary variables, the stiffness matrix of a discrete structural model can be formed globally as in the case of the matrix displacement method, or locally by considering the stiffness contributions of individual elements. This latter procedure is generally known as the direct stiffness method. The direct stiffness procedure allows the assembly of stiffness contributions from a finite number of elements that are used to model any complex structure. This is also the procedure used in a more generalized method known as the Finite Element Method

## 1. Structural System

Structural analysis is a process to analyze a structural system in order to predict the

responses of the real structure under the excitation of expected loading and external environment during the service life of the structure. The purpose of a structural analysis is to ensure the adequacy of the design from the view point of safety and serviceability of the structure. The process of structural analysis in relation to other processes is depicted in Figure 1.

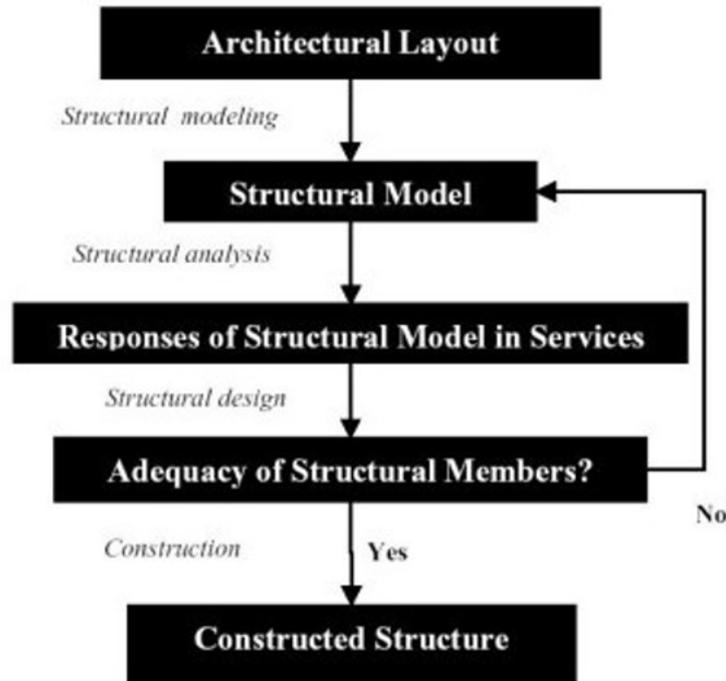


Figure 1. Role of structural analysis in the design process of a structure.

A structural system normally consists of three essential components as illustrated in Figure 2: (a) the structural model; (b) the prescribed excitations; and (c) the structural responses as the result of the analysis process. In all cases, a structure must be idealized by a mathematical model so that its behaviors can be determined by solving a set of mathematical equations.

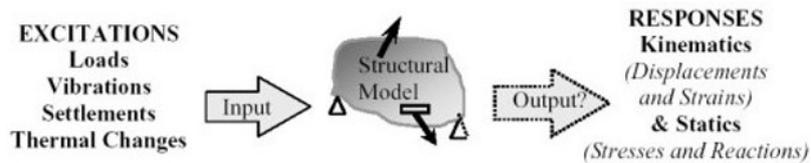
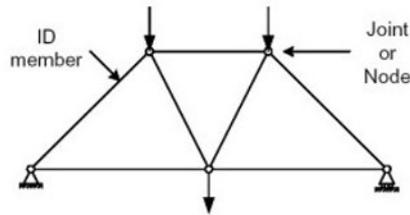


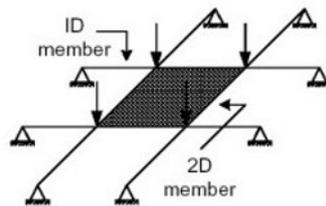
Figure 2. Definition of a structural system

A structural system can be one-dimensional, two-dimensional or three-dimensional depending on the space dimension of the loadings and the types of structural responses that are of interest to the designer. Although any real-world structure is strictly three-dimensional, for the purpose of simplification and focus, one can recognize a specific pattern of loading under which the key structural responses will remain in just one or two-dimensional space. Some examples of 2D and 3D structural systems are shown in Figure 3.

*Planar (2D) structures subjected to in-plane loads:*



*Planar (2D) structures subjected to loads normal to the plane:*



*General 3D structures:*

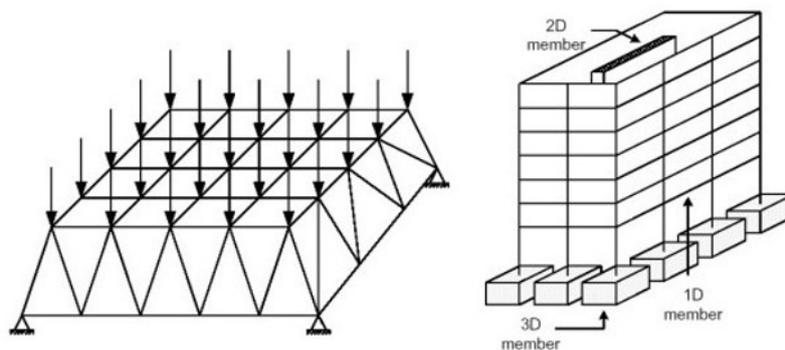


Figure 3. Some examples of one, two and three-dimensional structural models.

## 2. Structural Modeling

A structural (mathematical) model can be defined as an assembly of structural members (elements) interconnected at the boundaries (surfaces, lines, joints). Thus, a structural model consists of three basic components namely, (a) structural members, (b) joints (nodes, connecting edges or surfaces) and (c) boundary conditions.

**(a) Structural members:** Structural members can be one-dimensional (1D) members (beams, bars, cables etc.), 2D members (planes, membranes, plates, shells etc.) or in the most general case 3D solids.

**(b) Joints:** For one-dimensional members, a joint can be rigid joint, deformable joint or pinned joint, as shown in Figure 4. In rigid joints, both static and kinematics variables are continuous across the joint. For pinned joints, continuity will be lost on rotation as well as bending moment. In between, the deformable joint, represented by a rotational spring, will carry over only a part the rotation from one member to its neighbor offset by the joint deformation under the effect of the bending moment.

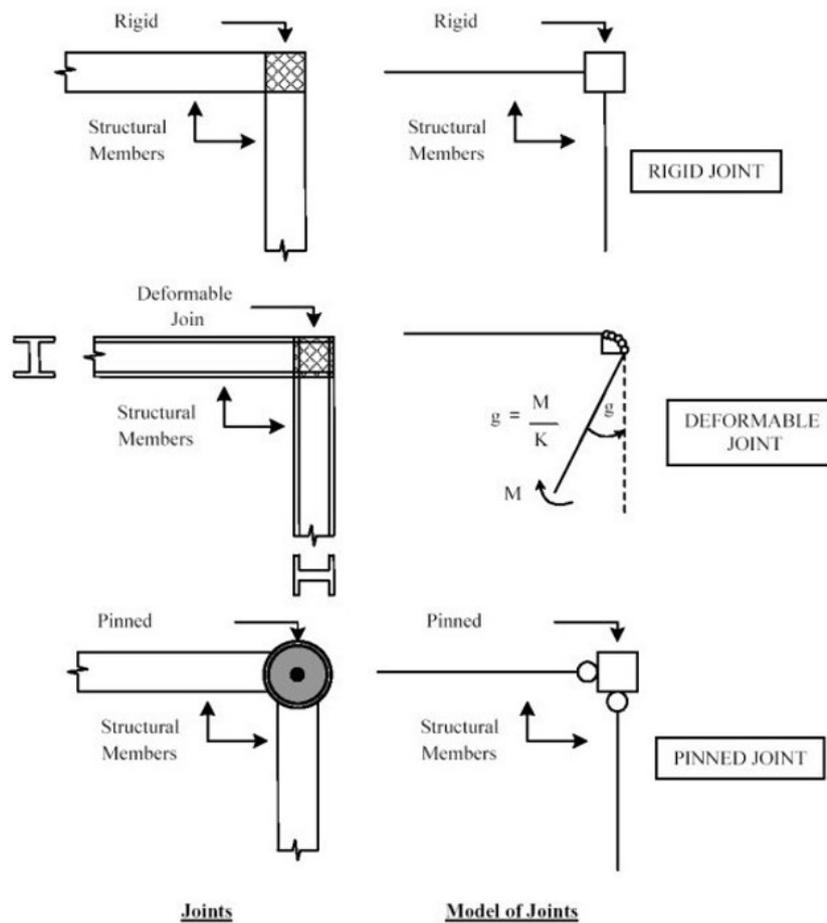


Figure 4. Typical joints between two 1D members.

(c) **Boundary conditions:** To serve its purposeful functions, structures are normally prevented from moving freely in space at certain points called *supports*. As shown in Figure 5, supports can be fully or partially restrained. In addition, fully restrained components of the support may be subjected to prescribed displacements such as ground settlements.

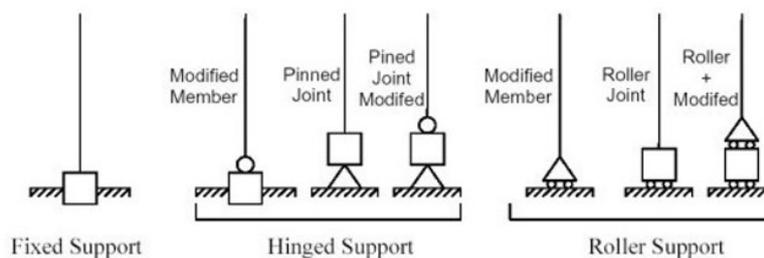


Figure 5. Boundary conditions of supports for 1D members.

### 3. Linearity of the Structural System

Assumptions are usually observed in order for the structural system to be treated as linear:

(a) The displacement of the structure is so insignificant that under the applied loads, the deformed configuration can be approximated by the un-deformed configuration in satisfying the equilibrium equations.

(b) The structural deformation is so small that the relationship between strain and displacement remains linear.

(c) For small deformation, the stress-strain relationship of all structural members falls in the range of Hooke's law, i.e., it is linear elastic, isotropic and homogeneous.

As a result of (a), (b) and (c), the overall structural system becomes a linear problem; consequently the principle of superposition holds.

#### 4. Definition of Kinematics

##### a) *Motions*

As shown in Figure 6, the motion of any particle in a body is a time parameter family of its configurations, given mathematically by

$$\hat{x} = \hat{x}(\tilde{P}, t) \quad (1)$$

where  $\hat{x}$  is a time function of a particle,  $P$ , in the body. During a motion, if relative positions of all particles remain the same as the original configuration, the motion is called a "rigid-body motion".

##### b) *Displacement*

Displacement of a particle is defined as a vector from its reference position to the new position due to the motion. If  $P_0$  is taken as the reference position of  $P$  at  $t = t_0$ , then the displacement of  $P$  at time  $t = t_1$  is

$$u(\tilde{P}, t_1) = \hat{x}(\tilde{P}, t_1) - \hat{x}(\tilde{P}, t_0) \quad (2)$$

##### c) *Deformation*

The quantitative measurement of deformation of a body can be presented in many forms:

(1) Displacement gradient matrix  $\nabla u$

$$\nabla u = [u_{i,j}] = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \quad (3a)$$

Note that this matrix is not symmetric.

(2) Deformation gradient matrix  $\tilde{F}$

$$\tilde{F} = \tilde{I} + \nabla \underline{u} \quad \text{or} \quad F_{ij} = \delta_{ij} + u_{i,j} \tag{3b}$$

where  $\tilde{I}$  is the identity matrix and  $\delta_{ij}$  is Kronecker delta. Like the displacement gradient matrix, the deformation gradient matrix is not symmetric.

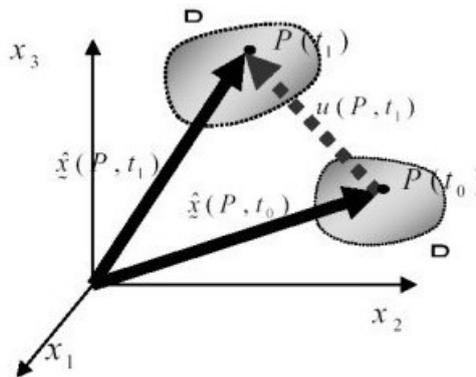


Figure 6. Motion of a body and position vectors of a particle.

(3) Infinitesimal strain tensor (for small strains)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{4}$$

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) & \frac{1}{2}(u_{1,3} + u_{3,1}) \\ & u_{2,2} & \frac{1}{2}(u_{2,3} + u_{3,2}) \\ (SYM) & & u_{3,3} \end{bmatrix} \tag{5}$$

The strain tensor is symmetric. Its symmetric shear strain components in the opposite off-diagonal positions can be combined to yield the *engineering shear strain*  $\gamma_{ij}$  as depicted in Figure 7 as,

$$\gamma_{ij} = \varepsilon_{ij} + \varepsilon_{ji} = u_{i,j} + u_{j,i} \tag{6}$$

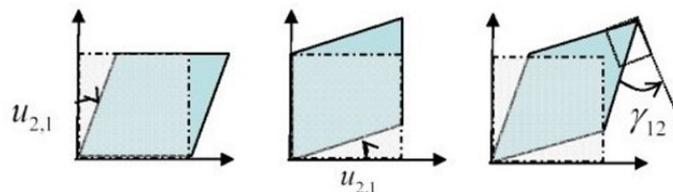


Figure 7. Engineering shear strain  $\gamma_{ij}$

Since  $\varepsilon$  is symmetric, there is no need to work with all the 9 components, strain tensor is often rewritten in a vector form as

$$\underline{\tilde{\varepsilon}} = [\varepsilon_{11} \ \varepsilon_{22} \ \varepsilon_{33} \ \gamma_{12} \ \gamma_{23} \ \gamma_{31}]^T \quad (7)$$

(4) Rotational Strain tensor, illustrated in Figure 8, is defined as

$$\omega_{ij} = \frac{1}{2}(u_{j,i} - u_{i,j}) \quad (8a)$$

or in matrix form:

$$\underline{\tilde{\omega}} = \begin{bmatrix} 0 & \frac{1}{2}(u_{2,1} - u_{1,2}) & \frac{1}{2}(u_{3,1} - u_{1,3}) \\ \frac{1}{2}(u_{1,2} - u_{2,1}) & 0 & \frac{1}{2}(u_{3,2} - u_{2,3}) \\ \frac{1}{2}(u_{1,3} - u_{3,1}) & \frac{1}{2}(u_{2,3} - u_{3,2}) & 0 \end{bmatrix} \quad (8b)$$

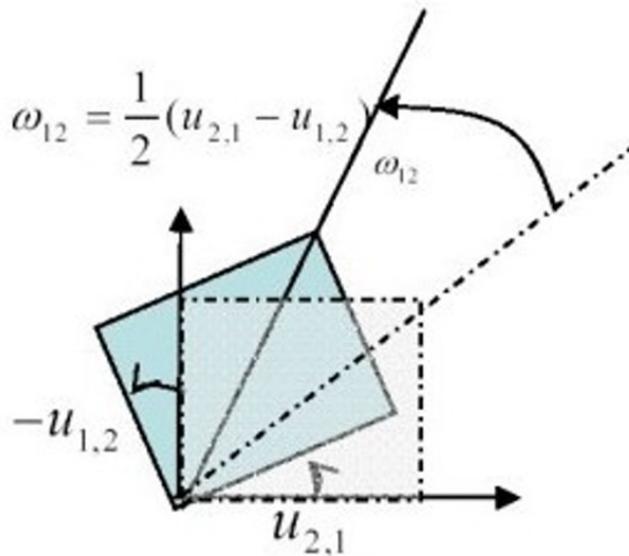


Figure 8. Illustration of a rotational strain  $\omega_{12}$

Some notes on the properties of  $\underline{\tilde{\varepsilon}}$  and  $\underline{\tilde{\omega}}$

a) One can easily show that

$$u_{i,j} = \varepsilon_{ij} - \omega_{ij} \quad (9)$$

and that a displacement gradient is decomposable into 2 parts  $\underline{\tilde{\varepsilon}}$  and  $-\underline{\tilde{\omega}}$ .

b) If  $u_{i,j} = u_{j,i}$  then  $\underline{\tilde{\omega}} = 0$  (pure deformation)

c) If  $u_{i,j} = -u_{j,i}$  then  $\underline{\tilde{\varepsilon}} = 0$  (pure rotation)

## 5. Definitions of Statics

Stress is defined as internal force per unit deformed area, distributed continuously within the domain of a continuum that is subjected to external applied forces. When the deformation is small, stress can be approximated by internal force per un-deformed area

based on its original configuration. This type of stress is represented by the *Cauchy stress tensor* in the form of

$$\underline{\sigma} = [\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad (10)$$

As illustrated in Figure 9, the two subscript indices  $i$  and  $j$  corresponding to  $x_i$  and  $x_j$  refer to the normal vector of the surface and the direction associated with the acting internal force.

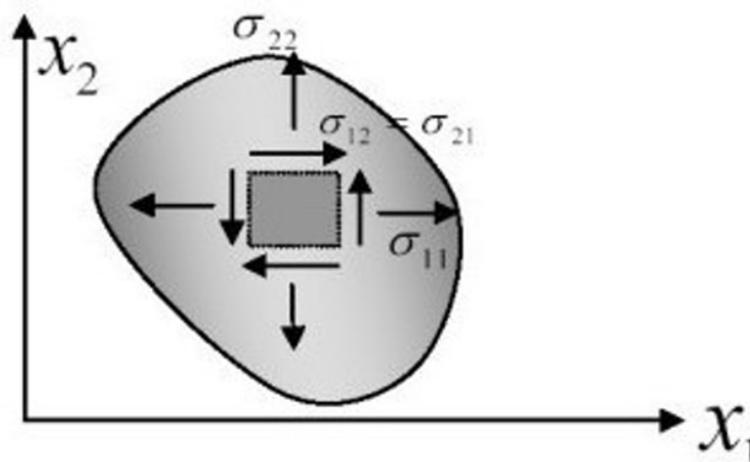


Figure 9. Definition of Cauchy Stress Tensor.

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### **Biographical Sketch**

**Worsak Kanok-Nukulchai** received his Ph.D. in Structural Engineering and Structural Mechanics from the University of California at Berkeley in 1978 before joining Asian Institute of Technology, where he is currently a professor of Civil Engineering. He has extensive research experiences in Computational Mechanics for the last 25 years and has published widely in the field. In 1997 he was admitted to be a member of the Royal Institute of Thailand. In 1999, he was recognized as the National Distinguished Researcher by Thailand's National Research Council. He is a Council Member of the International Association of Computational Mechanics, and the current Chairman of the International Steering Committee of a Regional Series of East-Asia Pacific Conferences on Structural Engineering and Construction.