MATHEMATICAL MODELING OF PETROLEUM EXTRACTION PROCESSES

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Contents

Summary

Yueming Cheng is the only author of this topic. She prepared all components of the manuscript, including text, tables, figures and bibliography etc.

1. Introduction

Mathematical modeling is the process of developing a mathematical model of physical phenomena in a system of the real world. As defined by Eykhoff (1974), a mathematical model is meant by “a representation of the essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in usable form”. In petroleum engineering area, mathematical modeling is to use mathematical language to describe a hydrocarbon reservoir system associated with the process of flow of fluids through subsurface porous media.

Recovery of petroleum hydrocarbons is a process, during which crude oil and/or natural gas are continuously extracted from hydrocarbon formations underground or under seafloor. This process typically accompanies a series of changes in pressure and hence

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changes in properties of rock and fluids contained, which can be modeled by a set of mathematical equations. Mathematical models are the theoretical foundation for optimization of hydrocarbon recovery scheme in oil/gas industry. They mainly include the diffusivity equation & analytical solutions, material balance equations, and decline-curve analysis, and are commonly used for predicting reservoir and production performance.

The diffusivity equation is a governing equation for fluid flow in porous media. It is most often used to model unsteady-state flow. Its analytical solutions under appropriate boundary and initial conditions have wide application in petroleum engineering field in addition to forecast production performance. For example, the analytical solutions provide the theoretical bases of pressure transient analysis technique, and employing analytical solutions is a cost-effective and readily available way to validate numerical simulators. Therefore, emphasis will be placed on constructing the diffusivity equation and its analytical solutions under corresponding conditions.

The diffusivity equation is a partial differential equation. The fundamental components to establish the diffusivity equation include the principle of mass conservation (a continuity equation), the law of conservation of momentum (an equation of fluid motion) and an equation of state (EOS). For some particular recovery methods, additional equations may be required. For example, thermal recovery of a heavy oil reservoir is a non-isothermal process, and the principle of energy conservation needs to be considered.

2. Mathematical Model of Single-Phase Flow – Slightly Compressible Fluids (Liquid)

2.1. Derivation of the Diffusivity Equation

The mathematical model, i.e., the diffusivity equation, is derived based on the continuity equation, an equation of fluid motion and an equation of state.

2.1.1. Continuity Equation

The continuity equation is a mathematical representation of the principle of mass conservation. We can obtain the continuity equation through either a differential method or an integral method.

(a) Differential method

Consider a small parallelepiped element in a porous medium, schematically illustrated in Figure 1. The flow of a fluid in the porous medium is along x, y and z directions.

The mass balance on the element can be written as

$$mass\ \ flow\ \ rate\ \ in - mass\ \ flow\ \ rate\ \ out = mass\ \ accumulation\ \ rate \qquad (1)$$

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The *x*-direction component of the mass flux, $q_{\mathrm{mi},x}$, into the element at *x* is

$$q_{\mathrm{mi},x} = \rho u_x \tag{2}$$

where ρ is the density of the fluid and u_x is velocity in x direction, respectively.

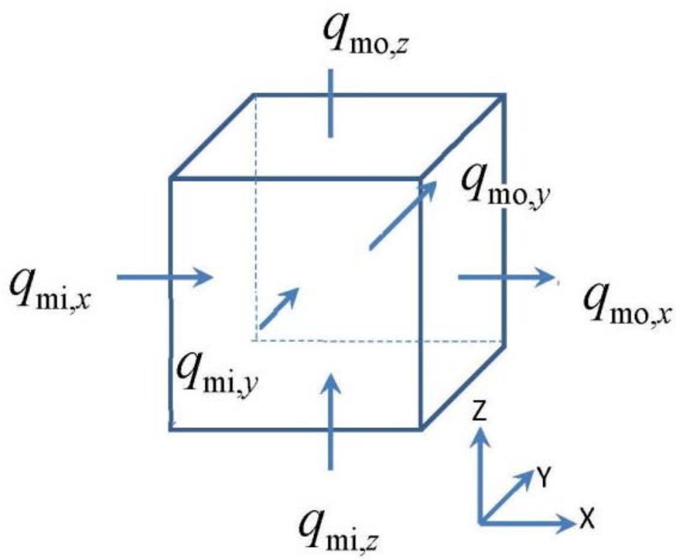

Figure 1. Illustration of parallelepiped element

The *x*-direction component of the mass flux, $q_{\mathrm{mo},x}$, out of the element at $x+\Delta x$ is

$$q_{\mathrm{mo},x} = \rho u_x + \frac{\partial(\rho u_x)}{\partial x}\Delta x \tag{3}$$

Hence, the difference of mass fluxes into and out of the element in x direction is

$$q_{\mathrm{mi},x} - q_{\mathrm{mo},x} = -\frac{\partial(\rho u_x)}{\partial x}\Delta x \tag{4}$$

and the difference of mass flow rate in and mass flow rate out of the element in x direction can be obtained as

$$(q_{\mathrm{mi},x} - q_{\mathrm{mo},x})\Delta y\Delta z = -\frac{\partial(\rho u_x)}{\partial x}\Delta x\Delta y\Delta z \tag{5}$$

where Δy and Δz are the sizes of the element in y direction and z direction, respectively.

The *y*-direction component of the mass flux, $q_{\mathrm{mi},y}$, into the element at y is

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$$q_{\mathrm{mi},y} = \rho u_y \tag{6}$$

where u_y is the fluid velocity in y direction.

The *y*-direction component of the mass flux, $q_{\mathrm{mo},y}$, out of the element at $y + \Delta y$ is

$$q_{\mathrm{mo},y} = \rho u_y + \frac{\partial(\rho u_y)}{\partial y}\Delta y \tag{7}$$

Hence, the difference of mass fluxes into and out of the element in y direction is

$$q_{\mathrm{mi},y} - q_{\mathrm{mo},y} = -\frac{\partial(\rho u_y)}{\partial y}\Delta y \tag{8}$$

and the difference of mass flow rate in and mass flow rate out of the element in y direction can be obtained as

$$(q_{\mathrm{mi},y} - q_{\mathrm{mo},y})\Delta x\Delta z = -\frac{\partial(\rho u_y)}{\partial y}\Delta x\Delta y\Delta z \tag{9}$$

The z-direction component of the mass flux, $q_{\mathrm{mi},z}$, into the element at z is

$$q_{\mathrm{mi},z} = \rho u_z \tag{10}$$

where u_z is the fluid velocity in z direction.

The z-direction component of the mass flux, $q_{\mathrm{mo},z}$, out of the element at $z + \Delta z$ is

$$q_{\mathrm{mo},z} = \rho v_z + \frac{\partial(\rho u_z)}{\partial z}\Delta z \tag{11}$$

Hence, the difference of mass fluxes into and out of the element in z direction is

$$q_{\mathrm{mi},z} - q_{\mathrm{mo},z} = -\frac{\partial(\rho u_z)}{\partial z}\Delta z \tag{12}$$

and the difference of mass flow rate in and mass flow rate out of the element in z direction can be obtained as

$$(q_{\mathrm{mi},z} - q_{\mathrm{mo},z})\Delta x\Delta y = -\frac{\partial(\rho u_z)}{\partial z}\Delta x\Delta y\Delta z \tag{13}$$

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Therefore, the left-hand side of Eq. (1), i.e. the total difference of mass flow rate into and mass flow rate out of the element in x, y and z all directions, is

$$
\begin{aligned}
&(q_{\mathrm{mi},x} - q_{\mathrm{mo},x})\Delta y\Delta z + (q_{\mathrm{mi},y} - q_{\mathrm{mo},y})\Delta x\Delta z + (q_{\mathrm{mi},z} - q_{\mathrm{mo},z})\Delta x\Delta y \\
&= -\left[\frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z}\right]\Delta x\Delta y\Delta z
\end{aligned}
\tag{14}
$$

The pore volume in the element is

$$V_{\mathrm{p}} = \varphi\Delta x\Delta y\Delta z \tag{15}$$

Therefore, the mass in the element is

$$m = \rho\phi\Delta x\Delta y\Delta z \tag{16}$$

The rate of change in mass with time in the element is given

$$\frac{dm}{dt} = \frac{\partial(\rho\phi)}{\partial t}\Delta x\Delta y\Delta z \tag{17}$$

The right-hand side of Eq. (1), i.e., the rate of mass accumulation in the element, is equal to the rate of change in mass with time in Eq. (17).

Combining Eqs. (1), (14) and (17), we have

$$-\left[\frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z}\right]\Delta x\Delta y\Delta z = \frac{\partial(\rho\phi)}{\partial t}\Delta x\Delta y\Delta z \tag{18}$$

By simplifying Eq. (18), the continuity equation is given in the following form.

$$\frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} = -\frac{\partial(\rho\phi)}{\partial t} \tag{19}$$

or

$$\nabla \bullet (\rho\mathbf{u}) = -\frac{\partial(\rho\phi)}{\partial t} \tag{20}$$

where ∇ is the del operator used for specifying the divergence of the vector $\mathbf{u}$.

(b) Integral method

Consider an arbitrary volumetric part of a porous medium, schematically illustrated in Figure 2. Its entire volume is V (a subset in 3D porous medium) with a surface area of S.

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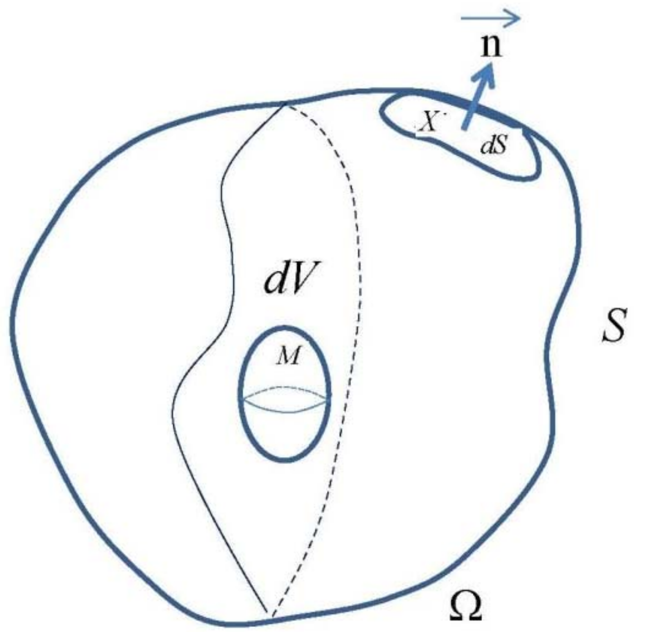

Figure 2. Arbitrary element in three dimensional space

The mass balance on the volume V can be written as

$$mass\ \ flow\ \ rate\ \ out = mass\ \ loss\ \ rate \tag{21}$$

Take a small element, ds, on the surface. The fluid mass flow rate, q_{m}, out of the surface element (ds) is

$$dq_{\mathrm{m}} = (\rho\mathbf{u} \bullet \mathbf{n})ds \tag{22}$$

where $\mathbf{n}$ represents the normal vector of ds and $\mathbf{u} \cdot \mathbf{n}$ stands for the normal component of $\mathbf{u}$.

Then the left-hand side of Eq. (21), i.e. the total mass flow rate out of the volume V can be obtained by integration over the entire surface (S).

$$q_{\mathrm{mo}} = \oiint_S (\rho\mathbf{u} \bullet \mathbf{n})ds \tag{23}$$

Take a small volume element, dV, anywhere in the volume V. We have the mass of fluid in dV as

$$dm = \rho\phi dV \tag{24}$$

Then, the total mass of fluid in the volume V is given by integration over the entire volume (V).

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$$m = \iiint_V \rho\phi dV \tag{25}$$

The right-hand side of Eq. (21), i.e. the rate of mass loss out of the volume V , is

$$\frac{dm}{dt} = -\iiint_V \frac{\partial(\rho\phi)}{\partial t} dV \tag{26}$$

Combining Eqs. (21), (23) and (26), then we have

$$\oiint_S (\rho\mathbf{u} \bullet \mathbf{n}) ds = -\iiint_V \frac{\partial(\rho\phi)}{\partial t} dV \tag{27}$$

From Gauss's theorem, the left-hand side of Eq. (27) can be written as

$$\oiint_S (\rho\mathbf{u} \bullet \mathbf{n}) ds = \iiint_V \nabla \bullet (\rho\mathbf{u}) dV \tag{28}$$

Substituting Eq. (28) into Eq. (27), we have

$$\iiint_V \nabla \bullet (\rho\mathbf{u}) dV = -\iiint_V \frac{\partial(\rho\phi)}{\partial t} dV \tag{29}$$

Assume that the integrands are continuous function in the volume V. As dV approaches zero, the two integrands in Eq. (29) must be equal. Therefore, the final form of continuity equation is given as

$$\nabla \bullet (\rho\mathbf{u}) = -\frac{\partial(\rho\phi)}{\partial t} \tag{30}$$

Eq. (30) derived from the integral method is exactly the same as the Eq. (20) derived from the differential method.

2.1.2. Equation of Motion

In general, the flow of a fluid through a porous medium is modeled by Darcy's law, which is an expression of conservation of momentum. Darcy's law is named after Henry Darcy, a French engineer, who formulated this law based on his experimental results (1856). It states that volumetric flow rate is proportional to the gradient of the potential.

Darcy's law is given by

$$q = -\frac{kA}{\mu} \nabla\Phi \tag{31}$$

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where q is volumetric flow rate, k is permeability of a porous medium, A is the cross-sectional area normal to the flow direction and μ is fluid viscosity. In Eq. (31), $\Delta\Phi$ is the gradient of flow potential defined as

$$\nabla\Phi = \nabla p + \rho g \nabla Z \tag{32}$$

where p is pressure, g is the gravitational acceleration and Z is elevation. The del operator (∇) is used to specify the gradients of Φ, p, and Z

A more general expression of Darcy's law is given by

$$\mathbf{u} = -\frac{\mathbf{k}}{\mu}\nabla\Phi \tag{33}$$

Where $\mathbf{u}$ is Darcy's velocity, defined by volumetric flow rate (q) divided by the cross-sectional area (A). Darcy's velocity is not an actual fluid velocity because it does not account for the space taken up by solid particles in a porous medium. The average fluid velocity, i.e., interstitial velocity, can be calculated as

$$\mathbf{v} = \frac{\mathbf{u}}{\phi} \tag{34}$$

In Eq. (33), $\mathbf{k}$ becomes the permeability tensor in 3D flow system with the following form.

$$\mathbf{k} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{35}$$

Because of symmetry of the permeability tensor (i.e., $k_{12}=k_{21}$), it has only six independent components.

Considering an isotropic reservoir so that components of the permeability tensor are the same in all directions and assuming that gravity effects are negligible, we obtain

$$\mathbf{u} = -\frac{k}{\mu}\nabla p \tag{36}$$

Therefore, components of Darcy's velocity along the coordinates of the flow system are given by

$$u_x = -\frac{k}{\mu}\frac{\partial p}{\partial x}; \quad u_y = -\frac{k}{\mu}\frac{\partial p}{\partial y}; \quad u_z = -\frac{k}{\mu}\frac{\partial p}{\partial z} \tag{37}$$

Darcy's law is valid for laminar flow.

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Biographical Sketch

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Station, U.S., M.S. from Research Institute of Petroleum Exploration and Development (RIPED) in Beijing, China and B.S. from Southwestern Petroleum University in Sichuan, China, all in Petroleum Engineering. She has more than 17 year experiences in petroleum industry and academics. Her expertise includes mathematical modeling of fluid flow and heat transfer in well-reservoir system, reservoir simulation, pressure transient analysis, reservoir characterization, hydraulic fracturing, unconventional gas recovery and uncertainty quantification. She is a SPE technical editor for the Journal of SPE Reservoir Engineering and Evaluation.

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