DESCRIPTION OF CONTINUOUS LINEAR TIME-INVARIANT SYSTEMS IN TIME DOMAIN

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Contents

Summary

This article introduces, with the aid of simple examples, some important descriptions of linear continuous time-invariant dynamical systems in the time domain. System descriptions such as differential equations, step response, and impulse response are discussed. Description in state space is also introduced.

1. Description by Differential Equations

The transfer behavior of linear continuous systems can be described by linear differential equations. Lumped parameter systems are described by ordinary differential equations, while distributed parameter system are modeled by partial differential equations. Apart from the approach of experimental identification, system models are derived on the basis of physical principles. In electrical systems we make use of the basic laws such as Kirchhoff's laws, the Ohm's law, the laws of induction etc. (in networks i.e., system with lumped parameters). Likewise we employ Maxwell's equations in distributed parameter systems (e.g., fields). In mechanical systems, we use Newton's laws, force, moment, and torque balance principles as well as the principle of conservation of energy, while in thermal systems the principles involved are that of conservation of internal energy or enthalpy, heat transfer and heat flow, often in combination with the laws of fluid dynamics. In order to enable a control engineer to tackle a wide variety of systems, typical examples of systems in the three fields mentioned above have been chosen for illustration in the following.

1.1. Electrical Systems

In order to handle electrical networks, one requires Kirchhoff's laws:

1. The algebraic sum of all currents at a junction (node) is zero. That is, $\Sigma i_k=0$ at any node.
2. The algebraic sum of all voltages in a mesh is zero. That is, $\Sigma u_k=0$ in any mesh.

The development of differential equation description of a network will now be illustrated with the aid of an example shown in Figure 1.

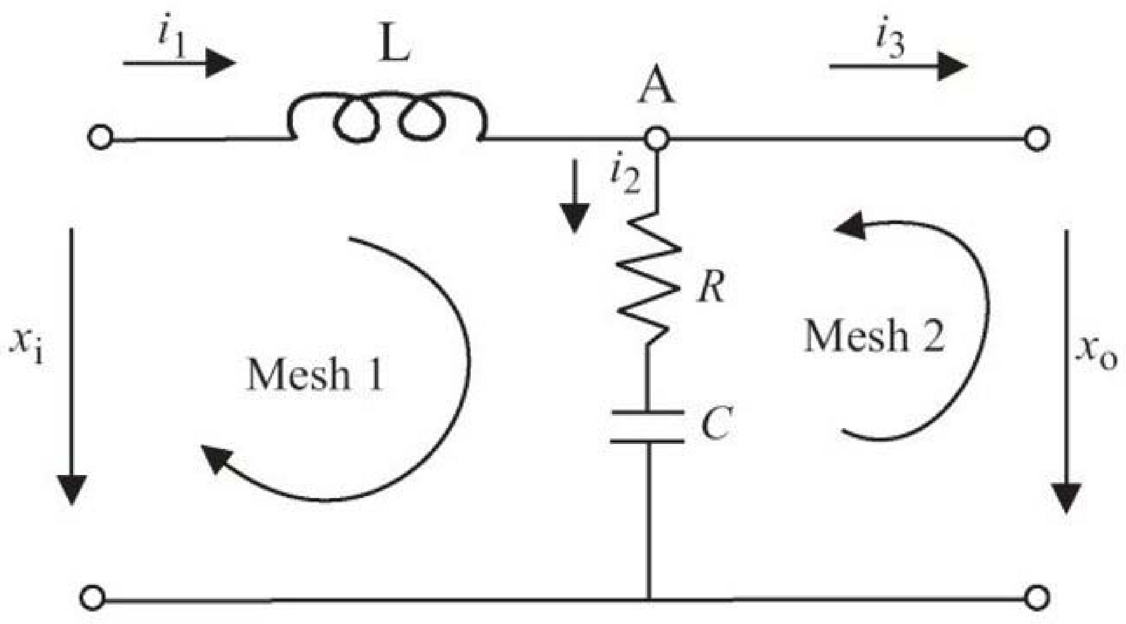

Figure 1. An electrical system

In this network R represents a resistance, C a capacitance, and L an inductance. The input and output variables, $x_i(t)$ and $x_0(t)$ respectively are the voltages at the ports. An initial charge on the capacitor is represented by the corresponding voltage $u_C(0)$. Let $i_1(0) = 0$.

Applying Krichhoff's laws:

For mesh 1:

$$x_i(t) = L\frac{\mathrm{d}i_1}{\mathrm{d}t} + Ri_2 + \frac{1}{C}\int_0^t i_2(\tau)\,\mathrm{d}\tau + u_\mathrm{C}(0). \tag{1}$$

For mesh 2:

$$x_0(t) = Ri_2 + \frac{1}{C}\int_0^t i_2(\tau)\,\mathrm{d}\tau + u_\mathrm{C}(0). \tag{2}$$

At node A:

$$i_1 - i_2 - i_3 = 0. \tag{3}$$

As the output port is not loaded (open), $i_3 = 0$. Therefore,

$$i_1 = i_2 = i. \tag{4}$$

Eqs. (1) and (2) lead to the relation:

$$x_\mathrm{i}(t) = L\frac{\mathrm{d}i_1}{\mathrm{d}t} + x_\mathrm{o}(t) \tag{5}$$

from which it follows that

$$i_1(t) = \frac{1}{L}\int_0^t [x_\mathrm{i}(\tau) - x_0(\tau)]\,\mathrm{d}\tau. \tag{6}$$

Using Eq. (4), i_1 is inserted in Eq. (2). This gives

$$x_\mathrm{o}(t) = R\frac{1}{L}\int_0^t [x_\mathrm{i}(\tau) - x_\mathrm{o}(\tau)]\,\mathrm{d}\tau + \frac{1}{CL}\int_0^t \int_0^{\tau_1} [x_\mathrm{i}(\tau_2) - x_\mathrm{o}(\tau_2)]\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + u_\mathrm{C}(0). \tag{7}$$

Differentiating Eq. (7) twice,

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$$\frac{\mathrm{d}^2 x_\mathrm{o}}{\mathrm{d}t^2} = \frac{R}{L}\left(\frac{\mathrm{d}x_\mathrm{i}}{\mathrm{d}t} - \frac{\mathrm{d}x_\mathrm{o}}{\mathrm{d}t}\right) + \frac{1}{CL}(x_\mathrm{i} - x_\mathrm{o}) \tag{8}$$

or

$$CL\frac{\mathrm{d}^2 x_\mathrm{o}}{\mathrm{d}t^2} + CR\frac{\mathrm{d}x_\mathrm{o}}{\mathrm{d}t} + x_\mathrm{o} = CR\frac{\mathrm{d}x_\mathrm{i}}{\mathrm{d}t} + x_\mathrm{i}\,. \tag{9}$$

If we denote $T_1 = RC$ and $T_2 = \sqrt{LC}$, we get for the given electric network finally the second order linear differential equation with constant coefficients:

$$T_2^2\frac{\mathrm{d}^2 x_\mathrm{o}}{\mathrm{d}t^2} + T_1\frac{\mathrm{d}x_\mathrm{o}}{\mathrm{d}t} + x_\mathrm{o} = x_\mathrm{i} + T_1\frac{\mathrm{d}x_\mathrm{i}}{\mathrm{d}t} \tag{10}$$

To determine $x_\mathrm{o}(t)$ uniquely the two initial conditions $x_\mathrm{o}(0)$ and $\dot{x}_\mathrm{o}(0)$ should be specified. The order of such a system is given by the number of independent energy storage elements (here L and C).

1.2. Mechanical Systems

In order to obtain the differential equations characterizing mechanical systems we require the following laws:

- Newton's laws of motion,
- Force, moment, and torque balance conditions, and
- Conservation of energy, linear momentum, and angular momentum.

As an example of a mechanical system consider the mass-spring-damper system shown in Figure 2.

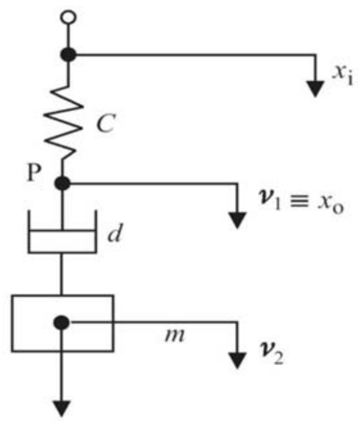

Figure 2. A mechanical system

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In this, c is the spring constant, d the damping constant, and m the mass. The variables v_1 (= x_o), v_2 and x_i denote the velocities and displacement at the points noted in the figure.

Newton's Law:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \sum F_i \quad , (F_i \text{ external forces})$$

in the present case gives

$$m\frac{\mathrm{d}v_2}{\mathrm{d}t} = d(v_1 - v_2) \tag{11}$$

Force balance at the point P (damping force = spring force) if the spring has no initial deflection at $t = 0$, gives

$$d(v_1 - v_2) = c\int_0^t [x_\mathrm{i}(\tau) - v_1(\tau)]\mathrm{d}\tau. \tag{12}$$

From Eqs. (11) and (12), we get

$$\frac{\mathrm{d}v_2}{\mathrm{d}t} = \frac{c}{m}\left[\int_0^t x_\mathrm{i}(\tau)\mathrm{d}\tau - \int_0^t v_1(\tau)\mathrm{d}\tau\right]. \tag{13}$$

Since v_1 is treated as the output variable x_0 of the system and the relation between v_1 and x_i is of interest, v_2 is eliminated. For this purpose Eq. (12) is differentiated with respect to time to give:

$$d\,\frac{\mathrm{d}v_1}{\mathrm{d}t} - \frac{\mathrm{d}v_2}{\mathrm{d}t} = c\left[x_\mathrm{i} - v_1\right]. \tag{14}$$

Inserting Eq. (13) in Eq. (14) and differentiating the result once more

$$\begin{aligned} &d\,\frac{\mathrm{d}^2 v_1}{\mathrm{d}t^2} - \frac{dc}{m}x_i + \frac{dc}{m}v_1 \\ &= c\frac{\mathrm{d}x_\mathrm{i}}{\mathrm{d}t} - c\frac{\mathrm{d}v_1}{\mathrm{d}t}. \end{aligned} \tag{15}$$

This is a second order linear differential equation with constant coefficients. Denoting

$$x_0 = \mathrm{v}_1,$$

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$T_1 = m/d$ and $T_2 = \sqrt{m/c}$, we get

$$T_2^2 \frac{\mathrm{d}^2 x_\mathrm{o}}{\mathrm{d}t^2} + T_1 \frac{\mathrm{d}x_\mathrm{o}}{\mathrm{d}t} + x_\mathrm{o} = x_\mathrm{i} + T_1 \frac{\mathrm{d}x_\mathrm{i}}{\mathrm{d}t}. \tag{16}$$

This equation possesses the same mathematical structure as that of the electrical network i.e., Eq.(10). The two systems are thus analogues of each other.

The analogy between mechanical systems made up of the basic elements *m*, *c*, and *d* and electrical systems with the basic elements *R*, *L*, and *C* can be generalized as shown in Figure 3.

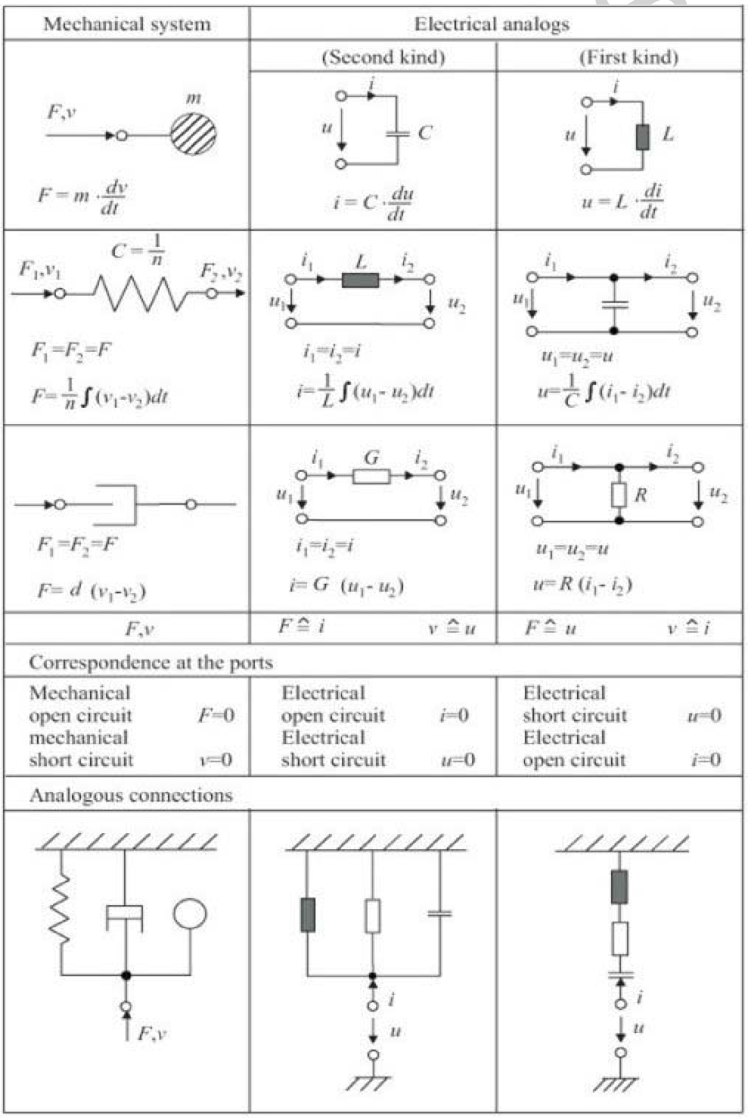

Figure 3. Electrical analogues of mechanical systems

The system shown in the first row and first column represents a mass under the action of force. It is analogous to a single port (two terminal electrical network). The systems in the next two rows in the first column on the other hand are analogous to two port (four terminal) networks. Analogy can be established between two systems by correspondence between the variables as effort and flow variables in the two systems. This can be done in two ways to obtain dual electrical analogues of mechanical systems:

1. Analogy of the first kind: Force ⇔voltage ($F \Leftrightarrow u$) and velocity ⇔ current ($v \Leftrightarrow i$)
2. Analogy of the second kind: Force ⇔ current ($F \Leftrightarrow i$) and velocity ⇔ voltage ($v \Leftrightarrow u$)

Although the two kinds of analogy lead to equations that have the same structure as the corresponding mechanical equation, it is the second kind that is often used, because, as shown in the bottom row of Figure 3, it preserves the topological structure (parallel connection) of the mechanical system. Notice that the first kind of analogy produces an electrical analogue that is a series circuit.

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Biographical Sketches

Heinz Unbehauen is Professor Emeritus at the Faculty of Electrical Engineering and Information Sciences at Ruhr-University, Bochum, Germany. He received the Dipl.-Ing. degree from the University of Stuttgart, Germany, in 1961 and the Dr.-Ing. and Dr.-Ing. habil. degrees in Automatic Control from the same university in 1964 and 1969, respectively. In 1969 he was awarded the title of Docent and in 1972, he was appointed as Professor of control engineering in the Department of Energy Systems at the University of Stuttgart. Since 1975, he has been Professor at Ruhr-University of Bochum, Faculty of Electrical Engineering, where he was head of the Control Engineering Laboratory until February 2001. He was Dean of his faculty in 1978/79. He was a Visiting Professor in Japan, India, China and the USA. He has authored and co-authored over 400 journal articles, conference papers and 7 books. He has delivered many invited lectures and special courses at universities and companies around the world. His main research interests are in the fields of system identification, adaptive control, robust control and process control of multivariable systems. He is Honorary Editor of IEE Proceedings on Control Theory and Application and System Science, Associate Editor of Automatica and serves on the Editorial Board of the International Journal of Adaptive Control and Signal Processing, Optimal Control Applications and Methods (OCAM) and Systems Science. He also served as associate editor of IEEE-Transactions on Circuits and Systems as well as Control-Theory and Advanced Technology (C-TAT). He is also an Honorary Professor of Tongji University Shanghai. He has been a consultant for many companies as well as for public organizations, e.g., UNIDO and UNESCO. He is member of several national and international professional organizations and Fellow of IEEE.

Ganti Prasada Rao was born in Seethanagaram, Andhra Pradesh, India, on August 25, 1942. He studied at the College of Engineering, Kakinada and received the B.E. degree in Electrical Engineering from Andhra University, Waltair, India in 1963, with first class and high honours. He received the M.Tech. (Control Systems Engineering) and Ph.D. degrees in Electrical Engineering in 1965 and 1970 respectively, both from the Indian Institute of Technology (IIT), Kharagpur, India. From July 1969 to October 1971, he was with the Department of Electrical Engineering, PSG College of Technology, Coimbatore, India as an Assistant Professor. In October 1971, he joined the Department of Electrical Engineering, IIT Kharagpur as an Assistant Professor and was a Professor there from May 1978 to June 1997. From May 1978 to August 1980, he was the Chairman of the Curriculum Development Cell (Electrical Engineering) established by the Government of India at IIT Kharagpur. From October 1975 to July 1976, he was with the Control Systems Centre, University of Manchester Institute of Science and Technology (UMIST), Manchester, England, as a Commonwealth Postdoctoral Research Fellow. During October 1981- November 1983, May-June 1985 and May-June 1991, he visited the Lehrstuhl fuer Elektrische Steuerung und Regelung, Ruhr-Universitaet Bochum, Germany as a Research Fellow of the Alexander von Humboldt Foundation. Since June 1992 he is on a visit to Abu Dhabi as Scientific Advisor to the Directorate of Power and Desalination Plants, Water and Electricity Department, Government of Abu Dhabi and the International Foundation for Water Science and Technology where he worked in the field of desalination plant control. He is presently a member of the UNESCO-EOLSS Joint Committee.

He has authored/coauthored four books: *Piecewise Constant Orthogonal Functions And Their Applications to Systems and Control, Identification of Continuous Dynamical Systems- The Poisson Moment Functional (PMF) Approach* (with D.C.Saha) , *General Hybrid Orthogonal Functions and their Applications in Systems and Control* (with A. Patra) all the three published by Springer in 1983, 1983 and 1996 respectively, *and Identification of Continuous Systems* (with H.Unbehauen) Published by North Holland in 1987. He is Co-Editor (with N.K.Sinha) of *Identification of Continuous Systems - Methodology and Computer Implementation,* Kluwer, 1991. He has co-authored (with A.Patra): *General Hybrid Orthogonal Functions and Their Applications in Systems and Control,* LNCIS-213, Springer, 1996. He has authored/coauthored over 150 research papers. He is on the Editorial Boards of *International Journal of Modeling and Simulation, Control Theory and Advanced Technology (C-TAT),*

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Systems Science (Poland), Systems Analysis Modeling and Simulation (SAMS) and *The Students' Journal of IETE(India).* He was Guest Editor of two Special Issues: one of *C-TAT on Identification and Adaptive Control - Continuous Time Approaches,* Vol.9, N0.1, March 1993, and *The Students' Journal of IETE on Control*, Vols. I&II, 1992-93. He is on the Honorary Editorial Advisory Board of *The American Biographical Research Institute*. He organized several invited sessions in IFAC Symposia on Identification and System Parameter Estimation, 1988, 1991, 1994 and World Congress 1993. He was a member of the IFAC Technical Committee on Modeling, Identification and Signal Processing in 1996. He was Chairman of the Technical Committee of the 1989 National Systems Conference in India. He is co-editor (with A. Sydow) of the book series "*Numerical Insights Series*" published by Gordon and Breach. He is a member of the Advisory Board of the Internal Study Group on Water and Energy Systems (ISGWES). Over the last several years, he has devoted himself to the development, from concept to completion, of the *Encyclopedia of Desalination and Water Resources* (DESWARE-online) and *Encyclopedia of Life support Systems* (EOLSS), two major publications of EOLSS Publishers, Oxford, UK.

He has received several academic awards including the IIT Kharagpur Silver Jubilee Research Award 1985, The Systems Society of India Award 1989, International Desalination Association Best Paper Award 1995 and Honorary Professorship of the East China University of Science and Technology, Shanghai. The International Foundation for Water Science and Technology has established the 'Systems and Information Laboratory' in the Electrical Engineering Department at the Indian Institute of Technology, Kharagpur, in his honor. He is listed in several biographic publications. Professor Rao is a Life Fellow of The Institution of Engineers (India), Fellow of The Institution of Electronics and Telecommunication Engineers (India), Fellow of IEEE (USA) and a Fellow of the Indian National Academy of Engineering.

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