DISCRETE-TIME, SAMPLED-DATA, DIGITAL CONTROL SYSTEMS, AND QUANTIZATION EFFECTS

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Summary

This article is concerned with the four subjects indicated in the title. We cover basic

properties of the discrete-time systems, the four well-known mathematical models for describing such systems (difference equations, transfer function, impulse response, and state-space equations), and their analysis on the basis of these four models. For the sampled-data systems we present the reason for discretizing continuous systems and give a brief description of D/A and A/D converters, as well as of the zero-order hold circuit. The problem of describing and analyzing sampled-data systems is also considered.

The digital control systems are briefly described, and their advantages over continuous control systems presented. Finally, the effects of quantization upon the performance of a digital controller are briefly set out.

1. Discrete-Time Systems

1.1. Introduction

The term *discrete-time system* covers systems that operate directly with discrete-time signals. In this case the input to the system, as well as the output from it, are both discrete-time signals (Figure 1). A well-known discrete-time system is the digital computer, wherein the signals $u(k)$ and $y(k)$ are number sequences. These types of system are described by difference equations, as opposed to continuous-time systems which are described by differential equations.

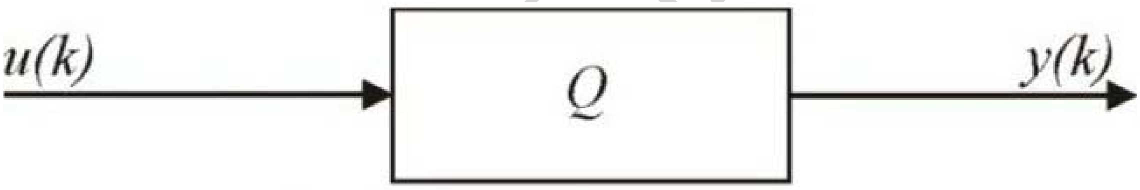

Figure 1. Block diagram of a discrete-time system

1.2. Properties of Discrete-Time Systems

From a mathematical point of view, a discrete-time system description implies the determination of a law that assigns an output sequence $y(k)$ to a given input sequence $u(k)$ (Figure 1). The specific law connecting the input and output sequences $u(k)$ and $y(k)$ constitutes the mathematical model of the discrete-time system. Symbolically, this relation can be written as follows:

$$y(k) = Q[u(k)]$$

where Q is a discrete operator.

Discrete-time systems have a number of properties, some of which are of special interest and are presented below.

1.2.1. Linearity

A discrete-time system is linear if the following relation

$$Q\left[c_1 u_1(k) + c_2 u_2(k)\right] = c_1 Q\left[u_1(k)\right] + c_2 Q\left[u_2(k)\right] = c_1 y_1(k) + c_2 y_2(k) \qquad (1)$$

holds true for every c_1, c_2, $u_1(k)$ and $u_2(k)$, where c_1, c_2 are constants, $y_1(k) = Q\left[u_1(k)\right]$ is the output of the system with input $u_1(k)$, and $y_2(k) = Q\left[u_2(k)\right]$ is the output of the system with input $u_2(k)$.

1.2.2. Time-Invariant System

A discrete-time system is time-invariant if the following

$$Q\left[u(k-k_0)\right] = y(k-k_0) \qquad (2)$$

holds true for every k_0. Eq. (2) shows that when the input to the system is shifted by k_0 units, the output of the system is also shifted by k_0 units.

1.2.3. Causality

A discrete-time system with zero initial conditions is termed causal if the output

$y(k) = 0$ for $k < k_0$, when the input u(k)=0 for $k < k_0$.

1.3. Description of Linear, Time-Invariant, Discrete-Time Systems

A linear, time-invariant, discrete-time system involves the following elements:

- summation units
- amplification units
- delay units

A block diagram with all three elements is shown in Figure 2. The delay unit is designated as z^{-1}, meaning that the output is identical to the input delayed by a time unit.

A linear, time-invariant, discrete-time system is described by a difference equation of the general form

$$\begin{aligned} y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) \\ = b_0 u(k) + b_1 u(k-1) + \cdots + b_m u(k-m) \end{aligned} \qquad (3)$$

From Eq. (1) one may easily derive the special case of the first-order system

$$y(k)+a_1y(k-1)=b_0u(k)+b_1u(k-1) \tag{4a}$$

Similarly, one may derive the special case of the second-order discrete-time system

$$y(k)+a_1y(k-1)+a_2y(k-2)=b_0u(k)+b_1u(k-1)+b_2u(k-2) \tag{4b}$$

and so on. Obviously, Eqs. (4a) and (4b) are mathematical models describing discrete-time systems, and are represented as block diagrams in Figures 3 and 4, respectively.

There are many ways to describe discrete-time systems, as is also the case for continuous-time systems. The most popular ones are:

- the difference equation, as in Eqs (3), (4a), and (4b)
- the transfer function
- the impulse response or weight function
- the state-space equations

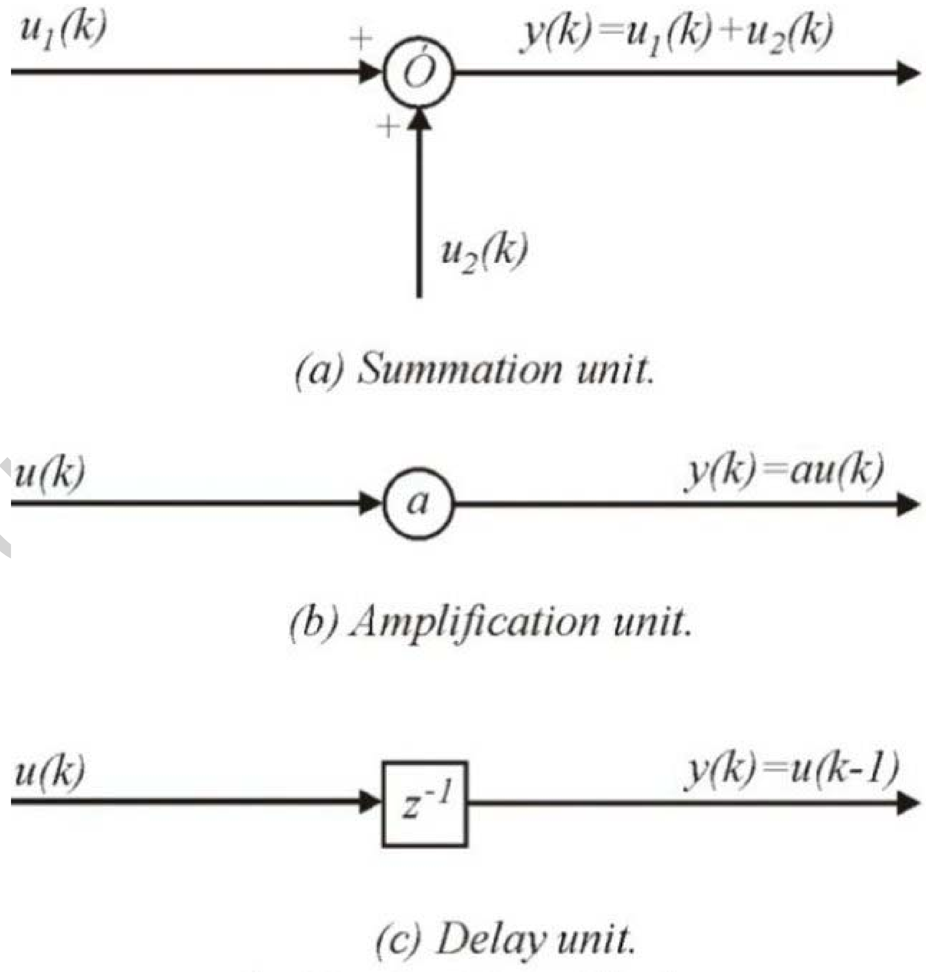

Figure 2. Summation, amplification, and delay units

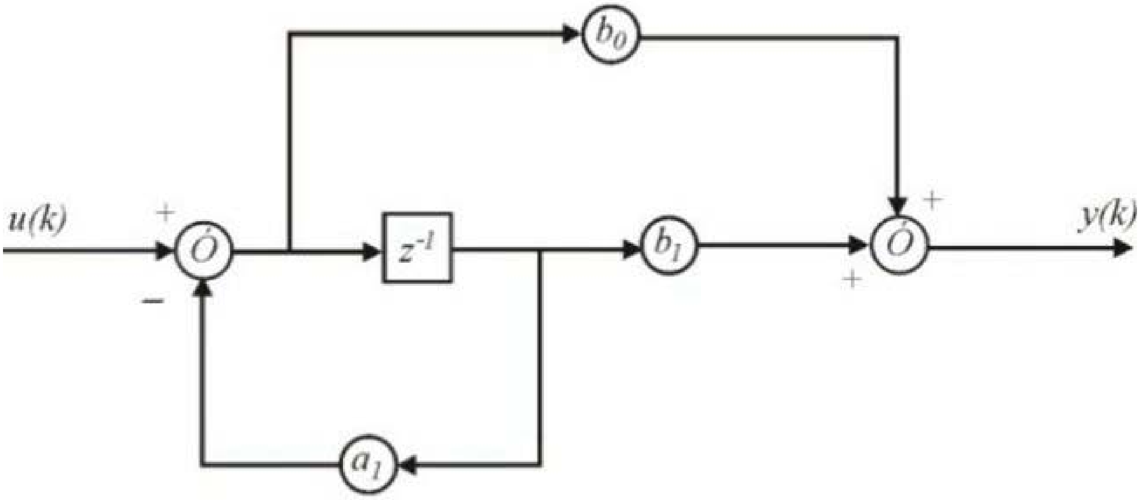

Figure 3. Block diagram of a first-order discrete-time system

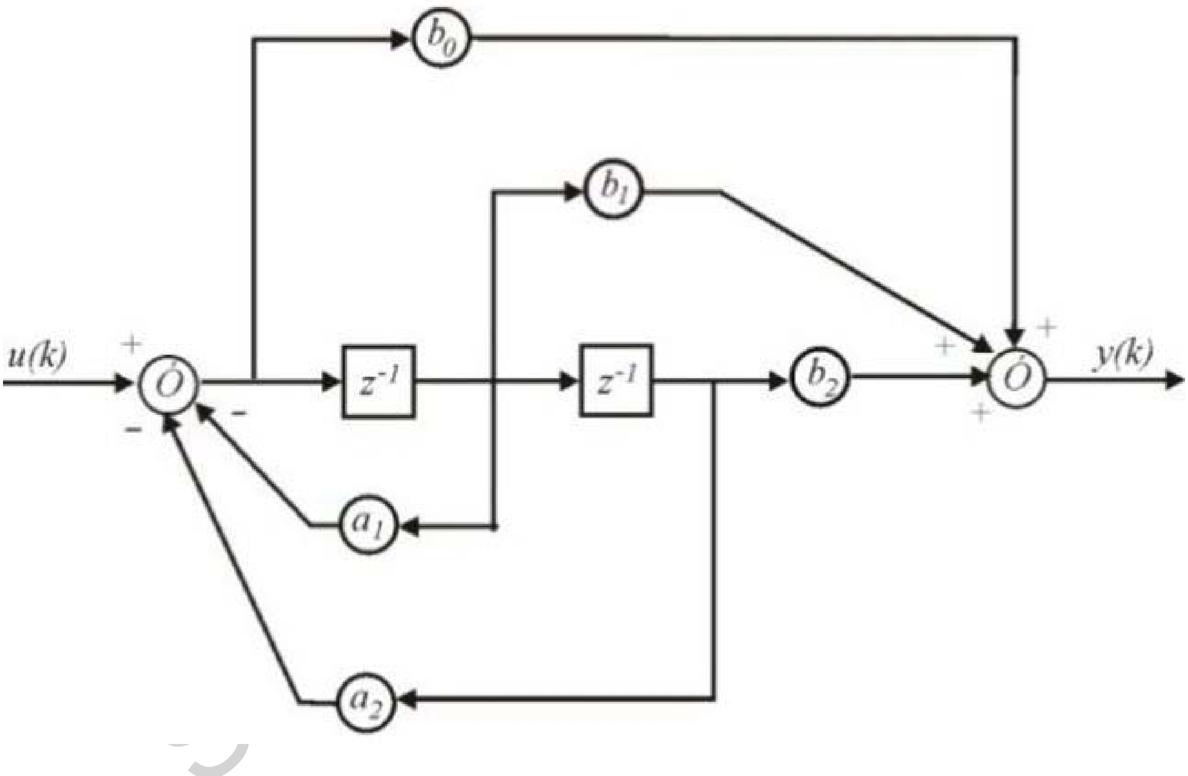

Figure 4. Block diagram of second-order discrete-time system

In the presentation of these four methods, certain similarities and dissimilarities between continuous-time and discrete-time systems will be revealed. There are three basic differences when going from continuous-time to discrete-time systems:

- differential equations are now difference equations,
- the Laplace transform gives way to the Z-transform, and
- the integration procedure is replaced by summation over k.

1.3.1. Difference Equations

The general form of a difference equation is given in Eq. (3) with initial conditions $y(-1), y(-2), ..., y(-n)$. The solution of Eq. (3) may be found either in the time domain (using methods similar to those for solving a differential equation in the time domain) or in the complex frequency or z-domain using the Z-transform.

1.3.2. Transfer Function

The *transfer function* of a discrete-time system is denoted by $H(z)$, and is defined as the ratio of the Z-transform of the output $y(k)$ divided by the Z-transform of the input u(k), under the condition that $u(k) = y(k) = 0$, for all negative values of k. That is

$$H(z) = \frac{Z[y(k)]}{Z[u(k)]} = \frac{Y(z)}{U(z)}, \text{ where } u(k) = y(k) = 0 \text{ for } k < 0 \tag{5}$$

where the Z-transform of a discrete function f(k) is defined as follows

$$F(z) = Z\left[f(k)\right] = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

and

$$F(z) = Z\left[f(k)\right] = \sum_{k=0}^{\infty} f(k) z^{-k}, \text{ when } f(k) = 0$$

The transfer function of a system described by the difference Eq. (4), with $u(k) = y(k) = 0$, for $k < 0$, is determined as follows: multiply both sides of Eq. (4) by the term z^{-k} and add for $k = 0,1,2,...,\infty$, to yield

$$\sum_{k=0}^{\infty} y(k) z^{-k} + a_1 \sum_{k=0}^{\infty} y(k-1) z^{-k}$$
$$+ a_2 \sum_{k=0}^{\infty} y(k-2) z^{-k} + ... + a_n \sum_{k=0}^{\infty} y(k-n) z^{-k}$$
$$= b_0 \sum_{k=0}^{\infty} u(k) z^{-k} + b_1 \sum_{k=0}^{\infty} y(k-1) z^{-k} + ... + b_m \sum_{k=0}^{\infty} y(k-m) z^{-k}$$

Using the Z-transform time-shifting property, and the assumption that $u(k)$ and $y(k)$ are zero for negative values of k, the equation above can be simplified as follows:

$$Y(z)+a_1z^{-1}Y(z)+\ldots+a_nz^{-n}Y(z)$$
$$=b_0U(z)+b_1z^{-1}U(z)+\ldots+b_mz^{-m}U(z)$$

where use was made of the definition of the Z-transform. Hence, using Eq. (5), we arrive at the following rational polynomial form for $H(z)$:

$$H(z)=\frac{Y(z)}{U(z)}=\frac{b_0+b_1z^{-1}+\ldots+b_mz^{-m}}{1+a_1z^{-1}+\ldots+a_nz^{-n}} \tag{6}$$

1.3.3. Impulse Response or Weight Function

The *impulse response* (or weight function) of a system is denoted by $h(k)$ and is defined as the output of a system when its input is the unit impulse sequence $\delta(k)$ under the constraint that the initial conditions $y(-1), y(-2),\ldots, y(-n)$ of the system are zero. The block diagram definition of the impulse response is shown in Figure 5. The transfer function $H(z)$ and the weight function $h(k)$ are related by the following equation

$$H(z)=Z[h(k)] \tag{7}$$

where $Z[f(k)]$ indicates the Z-transform of $f(k)$.

1.3.4. State-Space Equations

State-space equations, or simply *state equations*, are a set of first-order difference equations describing high-order systems, and have the form

$$\mathbf{x}(\mathrm{k}+1) = \mathbf{Ax}(\mathrm{k}) + \mathbf{Bu}(\mathrm{k}) \tag{8a}$$

$$\mathbf{y}(\mathrm{k}) = \mathbf{Cx}(\mathrm{k}) + \mathbf{Du}(\mathrm{k}) \tag{8b}$$

where $\mathbf{u}(k)\in R^m$, $\mathbf{x}(k)\in R^n$, and $\mathbf{y}(k)\in R^p$, are the input, state, and output vectors, respectively and **A**, **B**, **C**, and **D** are constant matrices of appropriate dimensions.

Let $\mathbf{Y}(z)=Z[\mathbf{y}(k)]$ and $\mathbf{U}(z)=Z[\mathbf{u}(k)]$. Then, the transfer function matrix $\mathbf{H}(z)$ of Eq. (8) is given by

$$\mathbf{H}(z)=\mathbf{C}[z\mathbf{I}_n-\mathbf{A}]^{-1}\mathbf{B}+\mathbf{D} \tag{9}$$

The impulse response matrix $\mathbf{H}(k)$ of (8) is given by

$$\mathbf{H}(k) = Z^{-1}[\mathbf{H}(z)] = \begin{cases} \mathbf{D}, & for \quad k=0 \\ \mathbf{CA}^{k-1}\mathbf{B} & for \quad k>0 \end{cases} \tag{10}$$

1.4. Analysis of Linear, Time-Invariant, Discrete-Time Systems

The problems involved in the analysis of linear, time-invariant, discrete-time systems will be treated using different methods, each corresponding to one of the four description models presented in Section 1.3.

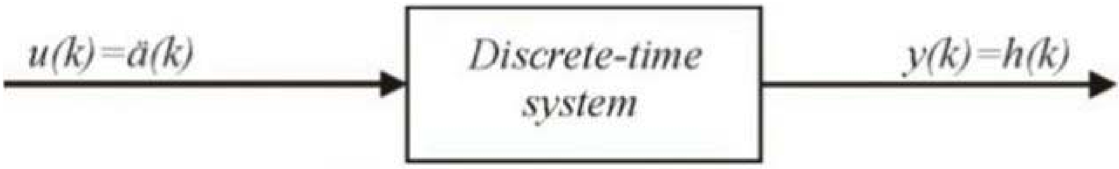

Figure 5. Block diagram definition of the impulse response

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Biographical Sketch

P.N. Paraskevopoulos was born in Doxa Arkadias, Greece, in 1941. He received his Bachelor’s (1964) and Master’s (1965) Degrees from Illinois Institute of Technology, and his Ph.D. from the University of Patras, Greece (1976). He has been Professor of Control in the Democritus University of Thrace (1977–1985) and in the National Technology University of Athens (1985 to date). He has published over 130 journal papers and 70 conference papers in the field of control engineering. He has written 10 books on control in Greek and two in English (*Digital Control Systems*, Prentice Hall, London, 1996; *Modern Control Engineering*, Marcel Dekker, New York, 2001). He is the founder and director of the Control Systems Laboratory, which is considered to be among the best of its kind in Europe. His current research and development interests are mainly in the following areas: system identification; system theory; controller design for linear and nonlinear multivariable systems; computer-controlled systems; optimal and algebraic control; adaptive and robust control; control of discrete-event systems; industrial applications, particularly in the processing industry (for example, refining, paper, cement, plastic, mill, aluminum, and food industries).