DESCRIPTION AND ANALYSIS OF DYNAMIC SYSTEMS IN STATE SPACE

Boris Lohmann
Institut für Automatisierungstechnik, Universität Bremen, Germany

Keywords: state space Description, state transformation, input-output relations, stability, eigenvalue, controllability, observability, poles and zeros, minimal realization, canonical form

Contents

Summary

This chapter begins by considering some fundamental properties of the state-variable representation of systems. These are intended, on the one hand, to clarify the relations with the frequency domain approach, and, on the other hand, are the basis for the controller and estimator design presented in the related chapters. Section 5 introduces two essential concepts of state space analysis: controllability and observability. Having defined these terms, four different criteria of controllability and observability will be discussed. Some remarks on poles, eigenvalues, zeros, and pole-zero cancellation complete the chapter.

1. Extraction of the State Space Representation from the Transfer Function *G*(*s*)

One way of getting the state space representation of a system has already been illustrated in *Design of State Space Controllers (Pole Placement) for SISO Systems* when the model of the inverted pendulum was derived; from the underlying physical equations and by eliminating some auxiliary variables, a state space model of the type

State differential equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$, (1)

Output equation: $y(t) = \mathbf{c}^{\mathrm{T}}\mathbf{x}(t)$, (2)

can often be determined in a straight forward manner. If, however, such a *theoretical modeling* is not possible but a system model in terms of a transfer function $G(s)$ is available (e.g. obtained from model *identification*, see *Modeling and Simulation of Dynamic Systems*, *Frequency Domain System Identification*, *Identification of Linear Systems in Time Domain*, *Identification of Nonlinear Systems*, *Bound-based Identification*, and *Practical Issues of System Identification*), then the question arises, how this model $G(s)$ can be converted into the state space representation (1), (2). A solution to this problem may be expected, since the reverse operation, the determination of the transfer function from the state equations, turns out to be a simple operation,

$$Y(s) = G(s)U(s), \quad \text{where} \quad G(s) = \mathbf{c}^{\mathrm{T}}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \tag{3}$$

(See *Design of State Space Controllers (Pole Placement) for SISO Systems*). The task of determining state Eqs. (1), (2) from a given transfer function $G(s)$ can be solved by different approaches. Three of them are presented here:

1.1. Solution 1: Control Canonical Form

Problem 1a): Let the system model be given by the transfer function $G(s)$,

$$Y(s) = \underbrace{\frac{1}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}}_{G(s)} \cdot U(s), \tag{4}$$

or equivalently by the ordinary differential equation

$$\overset{(n)}{y} + \ldots + a_1\dot{y} + a_0 y = u\,. \tag{5}$$

Find a state-variable representation (1), (2) with the same input-output relation $Y(s) = G(s)U(s)$.

Solution : The state variables are defined as

$$\begin{aligned} x_1 &= y, \\ x_2 &= \dot{y}, \\ &\vdots \\ x_n &= \overset{(n-1)}{y}. \end{aligned} \tag{6}$$

From this, n-1 simultaneous first-order differential equations are obtained immediately, and by substituting them into (5) a total of n differential equations is obtained,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \qquad\qquad \text{or} \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0 x_1 - a_1 x_2 \ldots - a_{n-1} x_n + u \end{aligned}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \ldots \\ 0 & \ddots & \ddots & 0 \\ & & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u\,, \tag{7}$$

either in a scalar representation (left) or in a matrix-vector representation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$ (right), which is the demanded state differential equation. The corresponding output equation $y = \mathbf{c}^{\mathrm{T}}\mathbf{x}$ is given by (6),

$$y = x_1 \quad \text{or} \quad y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x}\,. \tag{8}$$

Problem 1b): Extending problem 1a), the nominator of $G(s)$ is a polynomial now, i.e. the system is described by

$$Y(s) = \underbrace{\frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}}_{G(s)} U(s)\,, \tag{9}$$

or equivalently by the differential equation

$$\overset{(n)}{y} + \ldots + a_1\dot{y} + a_0 y = b_{n-1}\overset{(n-1)}{u} + \ldots + b_1\dot{u} + b_0 u\,. \tag{10}$$

Solution: First, (9) is rewritten as

$$Y(s) = b_0 \underbrace{\frac{1}{s^n + \ldots + a_0} U(s)}_{Y^*(s)} + b_1 s Y^*(s) + \ldots + b_{n-1}s^{n-1}Y^*(s)\,, \tag{11}$$

or equivalently in time domain,

$$y(t) = b_0 y^*(t) + b_1 \dot{y}^*(t) + \ldots + b_{n-1} \overset{(n-1)^*}{y}(t)\,. \tag{12}$$

For the representation of $Y^*(s)$ and $y^*(t)$ the solution to problem 1a) is used:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \ldots \\ 0 & \ddots & \ddots & 0 \\ & & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u\,, \tag{13}$$

$$y^* = x_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x}\,. \tag{14}$$

With $y^* = x_1$, $\dot{y}^* = \dot{x}_1 = x_2$, $\ddot{y}^* = \dot{x}_2 = x_3$,... the output $y(t)$ from (12) becomes

$$y = b_0 x_1 + \ldots + b_{n-1} x_n = \begin{bmatrix} b_0 & \cdots & b_{n-1} \end{bmatrix} \mathbf{x}\,. \tag{15}$$

Together, (13) and (15) are the result which is given an own name:

The state space model of the form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \ldots \\ 0 & \ddots & \ddots & 0 \\ & \ddots & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u\,, \tag{16}$$

$$y = \begin{bmatrix} b_0 & \cdots & b_{n-1} \end{bmatrix} \mathbf{x} \tag{17}$$

is called Control Canonical Form and has the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}\,. \tag{18}$$

1.2. Solution 2: Observer Canonical Form

In full analogy to the control canonical form, a so-called *observer canonical form* can be derived. This canonical form can simplify the observer design and is given here without further details:

The state space model of the form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & \dots & -a_0 \\ 1 & \ddots & \ddots & -a_1 \\ 0 & \ddots & 0 & \vdots \\ \dots & 0 & 1 & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} u\,, \tag{19}$$

$$y = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{x} \tag{20}$$

is called ***control canonical form*** and has the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}\,. \tag{21}$$

The term *control* is used because the design of state-feedback controllers will turn out to be particularly simple when starting from the control canonical form (See *Controller Design*).

Note that in the control canonical form each parameter of the transfer function only occurs once and that no calculation is required for writing down the state equations. In particular, the elements of the last row of the system matrix **A** represent the negative coefficients of the denominator of $G(s)$.

The special structure of the control canonical form is also expressed by its characteristic block diagram, shown in Figure 1.

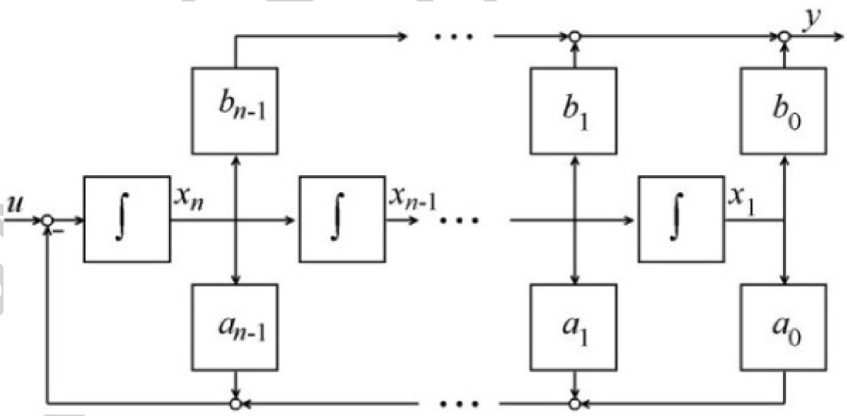

Figure 1: Block diagram of a system in control canonical form

Example: The model of the inverted pendulum with the transfer function

$$G(s) = \frac{-10}{s^4 - 16s^2} \tag{22}$$

can be represented in control canonical form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 16 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} -10 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}. \tag{23}$$

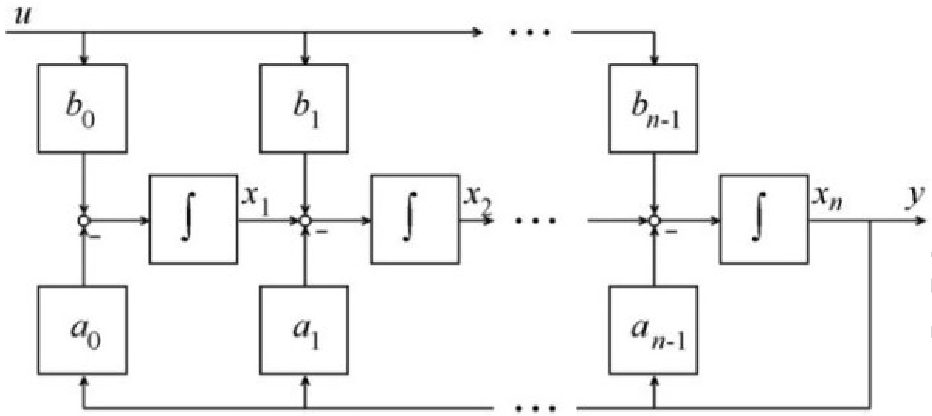

Figure 2: Block diagram of a system in observer canonical form

1.3. Solution 3: Modal Canonical Form (Diagonal and Jordan Canonical Form)

Problem 3a): Let the system model be given by the transfer function $G(s)$ from (9). Its poles (i.e. the zeros of its denominator) are assumed to be distinct (real or complex conjugate), so that a partial fraction expansion is given by

$$Y(s) = \left(\frac{r_1}{s - \lambda_1} + \ldots + \frac{r_n}{s - \lambda_n} \right) \cdot U(s). \tag{24}$$

Find a state space representation (1), (2) with the same input-output relation $Y(s) = G(s)U(s)$.

Solution: The state variables are defined as

$$X_1(s) = \frac{1}{s - \lambda_1} U(s), \ldots, \; X_n(s) = \frac{1}{s - \lambda_n} U(s), \tag{25}$$

or equivalently in time domain,

$$\dot{x}_1 = \lambda_1 x_1 + u, \ldots, \dot{x}_n = \lambda_n x_n + u. \tag{26}$$

These are in fact the n differential equations forming the state differential Eq. (1). The output equation is determined by substituting (25) into (24) and transforming into time domain:

$$Y(s) = r_1 X_1(s) + \ldots + r_n X_n(s) \qquad \Rightarrow \qquad y(t) = r_1 x_1 + \ldots + r_n x_n \,. \tag{27}$$

The state space model of the form

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{\dot{\mathbf{x}}(t)} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \ldots \\ 0 & \ddots & \\ \vdots & & \lambda_n \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{b}} u(t) \,, \tag{28}$$

$$y = \underbrace{\begin{bmatrix} r_1 & \ldots & r_n \end{bmatrix}}_{\mathbf{c}^T} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{x}(t)} \tag{29}$$

is called modal canonical form. Because A is a diagonal matrix, it is also called diagonal form of a system, and instead of A the letter Λ is used. The corresponding transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{r_1}{s - \lambda_1} + \ldots + \frac{r_n}{s - \lambda_n} \,. \tag{30}$$

Note that the *n* differential Eqs. (26) or (28) are fully decoupled: None of the *n* state variables is influenced by any other state variable. Therefore, these scalar differential equations can be examined separately, which is advantageous in many contexts. However, the poles λ_i and the residues r_i may be complex numbers which can somewhat complicate the handling. A further special feature of the modal canonical form is the fact that stability can easily be inspected by looking at the poles λ_i of $G(s)$ occurring as diagonal elements of **A**.

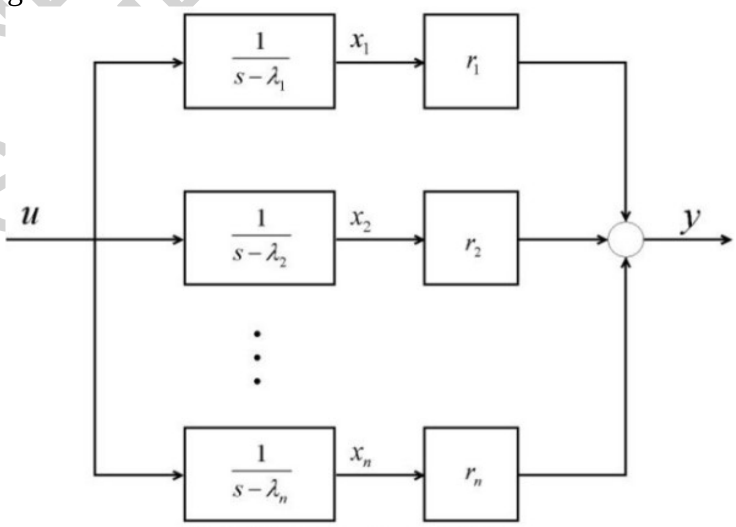

Figure 3: Block diagram of a system in diagonal form

Problem 3b): The assumption of distinct poles is dropped now, i.e. the transfer function $G(s)$ may have multiple poles. In particular, $G(s)$ is assumed to have a pole $\lambda_1 = \lambda_2 = ... = \lambda_k$ of magnitude k and distinct poles $\lambda_{k+1}, ..., \lambda_n$. Partial fraction expansion of $G(s)$ then results in

$$Y(s) = \left(\frac{r_1}{s-\lambda_1} + \frac{r_2}{(s-\lambda_1)^2} + ... + \frac{r_k}{(s-\lambda_1)^k} + \sum_{i=k+1}^{n} \frac{r_i}{s-\lambda_i} \right) U(s). \qquad (31)$$

Find a state space representation.

Solution: Again, each summand of (31) is assigned one state variable,

$$\begin{aligned} X_1(s) &= \frac{1}{s-\lambda_1} U(s) \\ X_2(s) &= \frac{1}{(s-\lambda_1)^2} U(s) = \frac{1}{s-\lambda_1} X_1(s) \\ &\vdots \\ X_k(s) &= \ldots = \frac{1}{s-\lambda_1} X_{k-1}(s) \\ X_{k+1}(s) &= \frac{1}{s-\lambda_{k+1}} U(s), \ \ldots, \ X_n(s) = \frac{1}{s-\lambda_n} U(s), \end{aligned} \qquad (32)$$

and again these equations are considered in time domain,

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + u \\ \dot{x}_2 &= \lambda_1 x_2 + x_1 \\ &\vdots \\ \dot{x}_k &= \lambda_1 x_k + x_{k-1} \\ \dot{x}_{k+1} &= \lambda_{k+1} x_{k+1} + u, \ \ldots, \ \dot{x}_n = \lambda_n x_n + u, \end{aligned} \qquad (33)$$

delivering the n scalar state differential equations. The output equation is

$$Y(s) = r_1 X_1(s) + ... + r_n X_n(s) \qquad \Rightarrow \qquad y(t) = r_1 x_1 + ... + r_n x_n. \qquad (34)$$

The state space model of the form

$$\dot{\mathbf{x}} = \underbrace{\left[\begin{array}{cccc|ccc} \lambda_1 & 0 & \cdots & 0 & & & \\ 1 & \lambda_1 & & \vdots & & 0 & \\ & \ddots & \ddots & 0 & & & \\ 0 & & 1 & \lambda_1 & & & \\ \hline & & & & \lambda_{k+1} & & 0 \\ & 0 & & & & \ddots & \\ & & & & 0 & & \lambda_n \end{array}\right]}_{\mathbf{A}} \mathbf{x} + \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ 1 \\ 1 \end{array}\right] u \quad , \tag{35}$$

$$y = \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix} \mathbf{x} \tag{36}$$

is called Jordan canonical form and is also a modal canonical form. Although the system matrix is only “close to diagonal”, the Jordan canonical form is sometimes referred to as diagonal form, and the letter Λ is used instead of A. The “close to diagonal” block in the left upper corner of A is called Jordan block and has dimensions (k,k). The corresponding transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{r_1}{s-\lambda_1} + \frac{r_2}{(s-\lambda_1)^2} + \ldots + \frac{r_k}{(s-\lambda_1)^k} + \sum_{i=k+1}^{n} \frac{r_i}{s-\lambda_i} \, . \tag{37}$$

If *several* multiple poles occur in $G(s)$ the steps shown above are adapted accordingly. The system matrix **A** will then comprise *several* Jordan blocks. The dimension of each Jordan block equals the magnitude of the corresponding pole.

Example: The transfer function of the balanced pendulum has a zero pole with magnitude two. The partial fraction expansion is

$$G(s) = \frac{-10}{s^4 - 16s^2} = \frac{0}{s} + \frac{5/8}{s^2} + \frac{-5/64}{s-4} + \frac{5/64}{s+4} \, . \tag{38}$$

Therefore the state space representation of the system is obtained in Jordan canonical form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 0 & 5/8 & -5/64 & 5/64 \end{bmatrix} \mathbf{x} \, . \tag{39}$$

-
-
-

Bibliography

Ackermann J. (1972). Der Entwurf linearer Regelungssysteme im Zustandsraum. *Regelungstechnik* **20**, 297-300. [Presents Ackermann's formula]

DeCarlo R.A. (1989). *Linear Systems*. Prentice Hall. [Includes an introduction to the description and analysis by state space methods].

Franklin G.F., Powell J.D. and Emami-Naeini A. (1994). *Feedback Control of Dynamic Systems*, 3rd edition. Addison-Wesley. [Textbook with good introduction to the key objectives of automatic control. Mathematical appendix].

Gilbert E.G., (1963). Controllability and Observability in Multivariable Control Systems. *J. Control, Ser. A*, **1**,. 128-151. [Includes the controllability/observability criteria presented].

Hautus M.L.J. (1969). Controllability and Observability Conditions of Linear Autonomous Systems. *Indagationes Mathematicae* **31**, 443-448. . [Includes the controllability/observability criteria presented].

Kaczorek T. (1992). *Linear Control Systems. Volume 1 and 2*. Research Studies Press. [Extensive theoretical work; includes linear differential algebraic systems and many details].

Kailath, T. (1980). *Linear Systems*. Prentice Hall. [A standard textbook on linear theory].

Kalman, R.E. (1960). A new Approach to Linear Filtering and Prediction Problems. *J. Basic Engineering* **85**, 394-400. [An initial work on state space methods].

Kalman R.E. (1960). *On the General Theory of Control Systems*. Proc. 1st Int. Congr. Autom. Control, Moscow, pp. 481-492. [An initial work on state space methods].

Kuo B.C. (1995). *Automatic Control Systems*, 7th edition. Prentice Hall. [Extensive textbook. Some details missing in favor of good readability].

Luenberger D.G. (1964). Observing the State of a Linear System. *IEEE Transactions on Military Electronics* **8**, 74-80. [Presents Luenberger's approach to state observers].

Luenberger D.G. (1966). Observers for Multivariable Systems. *IEEE Transactions on Automatic Control* **11**, 190-197. [Presents Luenberger's approach to state observers].

Luenberger D.G. (1971). An Introduction to Observers. *IEEE Transactions on Automatic Control* **16**, 596-602. [Presents Luenberger's approach to state observers].

MacFarlane A.G.J. and Karcanias N. (1976). Poles and Zeroes of linear multivariable systems: a survey of the algebraic, geometric and complex-variable theory. *Int. J. Control* **24**, 33-74. [Introduces the concept of invariant zeros from a state variable approach].

Shinners S.M. (1998). *Advanced Modern Control System Theory and Design*. John Wiley & Sons. [A recent textbook].

MATLAB applications: The following two web-addresses provide introductory examples on how to use the software package MATLAB for control system design purposes: http://tech.buffalostate.edu/ctm/ and http://www.ee.usyd.edu.au/tutorials_online/matlab/index.html

Biographical Sketch

Boris Lohmann received the Dipl.-Ing. and Dr.-Ing. degrees in electrical engineering from the Technical University of Karlsruhe, Germany, in 1987 and 1991 respectively. From 1987 to 1991 he was with the Fraunhofer Institut (IITB) and with the Institute of Control Systems, Karlsruhe, working in the fields of autonomous vehicles control and multi-variable state space design.

From 1991 to 1997 he was with AEG Electrocom Automation Systems in the development department for postal sorting machines, at last as the head of mechanical development. In 1994 he received the 'Habilitation' degree in the field of system dynamics and control from the Universität der Bundeswehr, Hamburg, for his results on model order reduction of nonlinear dynamic systems.

Since 1997 he has been full professor at the University of Bremen, Germany, and head of the Institute of Automation Systems. His fields of research include nonlinear multivariable control theory; system modeling, simplification, and simulation; and image-based control systems, with industrial applications in the fields of autonomous vehicle navigation, active noise reduction, error detection and fault diagnosis.