SECOND ORDER SYSTEMS

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Contents

Summary

A significant amount of the research into nonlinear differential equations in the nineteenth and early twentieth centuries done by mathematicians was devoted to second order differential equations. There were two major reasons for this, namely that the dynamics of many problems of practical interest could be approximated by these equations and secondly the phase plane approach allowed a graphical examination of their solutions. This section gives a brief overview of how the phase plane approach was developed for use in control system analysis and design.

The major contribution of control engineers was to consider use of the approach for nonlinear elements which could be defined by linear segmented characteristics, good approximations for many of the nonlinear effects in control systems, rather than continuous mathematical functions considered previously. A major advantage of the approach, as illustrated in this section, is that it can be used when more than one nonlinearity are present in the feedback loop.

1. Introduction

The dynamic equations representing many simple control systems, for example those of a position control system, may often be represented by second order nonlinear differential equations. Also in the early years of the development of control theory, from say 1930-1960, there was a major interest in these types of systems for the position control of radar antennas, guns and later radio telescopes. Further systems described by second order nonlinear differential equations representing problems found in nonlinear mechanics and electronic oscillations had been studied previously in the late nineteenth and early twentieth centuries by physicists using the phase plane method. It was therefore not surprising that much of the early work on nonlinear control used this approach. Control engineers did make significant contributions to this field since, whereas the earlier work had typically assumed nonlinearities defined by continuous mathematical functions, for control system analysis it was often more appropriate to approximate intrinsic nonlinearity, such as friction, or intentionally introduced nonlinearity, such as a relay, by linear segmented characteristics. The approach is still useful today because of the physical understanding it can provide and also because more than one nonlinearity can be considered.

2. Basic Principles

The formulation used in early work on second order systems was to assume a representation in terms of the two first order equations

$$\begin{aligned} \dot{x}_1 &= P(x_1,\ x_2) \\ \dot{x}_2 &= Q(x_1, x_2) \end{aligned} \tag{1}$$

Equilibrium, or singular points, occurs when

$$\dot{x}_1 = \dot{x}_2 = 0$$

and the slope of any solution curve, or trajectory, in the x_1 - x_2 state plane is

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{Q(x_1,\ x_2)}{P(x_1,\ x_2)} \tag{2}$$

The basic approach used was to determine the slope at a sufficient number of points in the state plane to allow a picture of the motion to be obtained starting from any initial conditions in the x_1 - x_2 state plane. Typically a second order nonlinear differential equation representing a control system with smooth nonlinearity can be written as

$$\ddot{x} + f(x, \dot{x}) = 0$$

and if this is rearranged as two first order equations, choosing the phase variables as the state variables, that is $x_1 = x,\ x_2 = \dot{x}$, then it can be written as

$$\begin{aligned} \dot{x}_1 &= \dot{x}_2 \\ \dot{x}_2 &= -f(x_1, x_2) \end{aligned} \tag{3}$$

which is a special case of Eq. (2). A variety of procedures have been proposed for sketching phase plane trajectories for Eq. (3). A complete plot showing trajectory motions throughout the entire phase plane from different initial conditions is known as a phase portrait. Today’s simulation methods enable phase plane trajectories to be easily displayed and they can often more clearly provide a picture of the system behavior than time response plots for x_1 and x_2.

Many investigations using the phase plane technique were concerned with the possibility of the nonlinear differential equations having limit cycle solutions. When a limit cycle exists this results in a closed trajectory in the phase plane and typical of such investigations was the work of Van der Pol on oscillators. He considered the equation

$$\ddot{x}-\mu(1-x^2)\dot{x}+x=0 \qquad (4)$$

where μ is a positive constant. The phase plane form of this equation can be written as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -f(x_1,\ x_2) = \mu(1-x_1^2)x_2 - x_1$$

The slope of a trajectory in the phase plane is

$$\frac{dx_2}{dx_1}=\frac{\dot{x}_2}{\dot{x}_1}=\frac{\mu(1-x_1^2)x_2-x_1}{x_2} \qquad (5)$$

which is only singular (that is at an equilibrium point), when the right hand side of Eq. (5) is 0/0, that is $x_1 = x_2 = 0$.

The form of the singular point, which is obtained from linearization of the equation at the origin, depends upon μ, being an unstable focus for $\mu < 2$ and an unstable node for $\mu > 2$. All phase plane trajectories have a slope of r when they intersect the curve

$$rx_2=\mu(1-x_1^2)x_2-x_1 \qquad (6)$$

One way of sketching phase plane behavior is to draw a set of curves for various selected values of r in Eq. (6) and marking the trajectory slope r on the curves, a procedure known as the method of isoclines. Figure 1 shows results sketched using isoclines for this equation with $\mu = 0.2$ and 5.0, respectively.

If a harmonic balance approach (see Chapter Describing Function Method) is used for this equation one assumes $x_1 = a\sin\omega t$ and $x_2=\dot{x}_1=a\omega\cos\omega t$. Rewriting Eq. (4) as

$$\dot{x}_2+x_1=\mu(1-x_1^2)x_2$$

then substituting for x_1 and x_2 and neglecting frequencies higher than ω in the right hand side gives

$$-a\omega^2\sin\omega t+a\sin\omega t=\mu(a\omega\cos\omega t-\frac{a^3\omega}{4}\cos\omega t)\cdot$$

For this to be true requires

$\omega = 1$
and

$a\omega = a^3\omega / 4$,

that is a limit cycle solution, which is independent of μ, with frequency 1 rad/s and amplitudes 2 for x_1 and x_2. Thus from Figure 1 it can be seen that this result is only reasonable for small values of μ.

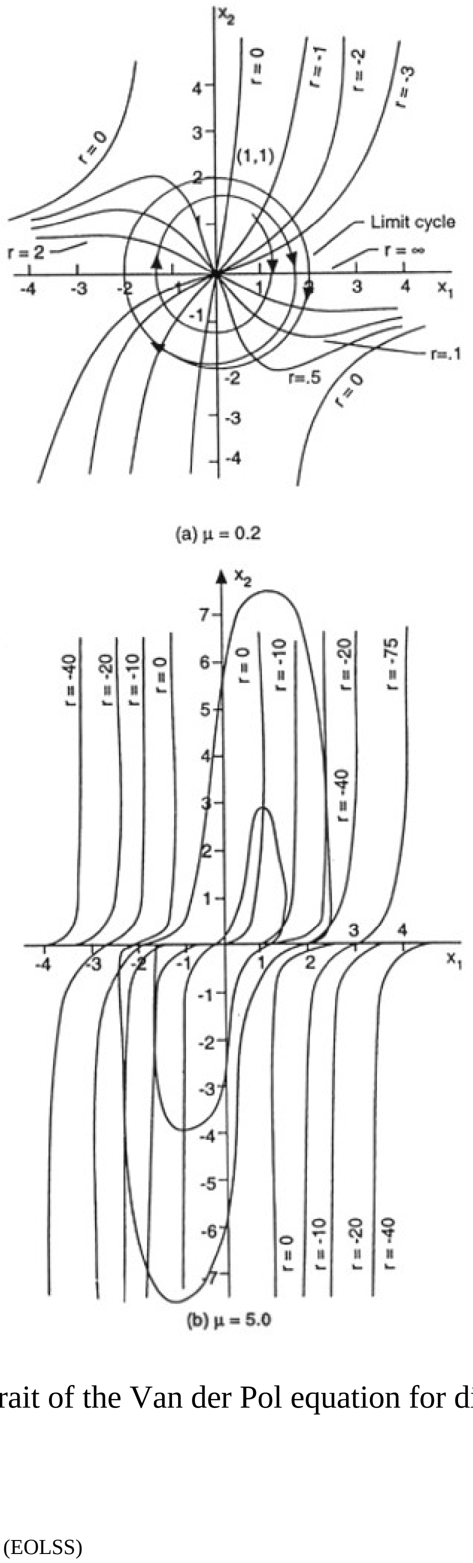

Figure 1: Phase portrait of the Van der Pol equation for different value of μ

Many nonlinear effects in control systems, such as saturation, friction etc., are best approximated by linear segmented characteristics rather than continuous mathematical functions. This is an advantage for study using the phase plane approach since it results in a phase plane divided up into different regions but with different linear differential equations describing the motion in each region. To illustrate these basic concepts three examples are considered in the next section.

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Biographical Sketch

Derek Atherton was born in Bradford, England on 21 April 1934. He has a B.Eng from Sheffield University and Ph.D and D.Sc from Manchester University. He taught at Manchester University, and McMaster University and the University of New Brunswick in Canada before taking up the appointment of Professor of Control Engineering at the University of Sussex in 1980, where he currently has a part-time appointment. He has served on committees of the Science and Engineering Research Council, as President of the Institute of Measurement and Control in 1990 and President of the Control Systems Society of the Institute of Electrical and Electronic Engineers, USA in 1995, and also served for six years on the International Federation of Automatic Control (IFAC) Council. His major research interests are in nonlinear control theory, computer aided control system design, simulation and target tracking. He has written three books, one of which is jointly authored, and published over 300 papers.