MINIMUM VARIANCE CONTROL

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Keywords: minimum variance control, prediction, stochastic processes.

Contents

Summary

A modern view of minimum variance control is presented by generalizing the notion of prediction and considering both discrete-time and continuous-time formulations.
The relation to other control design methods is also presented.

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1. Introduction

If it were possible to predict the future, it would be possible to plan ahead and adjust current actions to give desired future affects. Of course, unforeseen circumstances preclude exact prediction; but an approximate prediction allows better planning than no prediction at all. This is the basic idea of minimum variance control: predict future system outputs and adjust the current control signal to give the future system output a desired value.

Minimum variance control was developed as an approach to the control of systems with time delay with particular application to the paper-making industry. It was developed in a discrete-time stochastic setting, but was extended over a number of years to include both deterministic and continuous-time formulations. Although initially orientated towards system with time delay by using the idea of prediction, it was seen to be a special case of a wider class of methods: *Generalized minimum variance control*. This generalized viewpoint is used here.

An important application of these methods is as a basis for *Self-tuning Control* (see *Self-tuning Control*).

Both, minimum-variance control, and its extension to generalized minimum variance control, are based on predicting a single time instant into the future. To some extent, this approach has been overtaken by Model-based Predictive Controllers (see *Model-based Predictive Controllers for Linear Systems*), which predict over a span of time into the future. However, they have the advantage of being simpler and, in some cases, they are as effective.

2. Prediction

Minimum-variance control and its extensions are based on the notion of *predicting* the future output of a dynamic system based on:

- a *model* of the dynamic *system*
- a *model* of the *disturbances* affecting the system
- current and past *measurements* of the system input and output.

There are two dichotomies in the form of the model (see *General Models of Dynamic Systems*):

1. continuous-time or discrete-time
2. transfer-function or state-space

and all combinations have been used in the literature. This article focuses on the *transfer-function* approach and gives both discrete-time and continuous-time versions.

2.1. Discrete-Time Model

Consider the discrete-time system (see *Discrete-Time*, *Sampled-Data*, *Digital Control Systems*, *Quantisation Effects*).

$$A(z)y_i = B(z)u_i + C(z)\zeta_i \tag{1}$$

where y_i, u_i, and ζ_i are the system output, input and disturbance process at the discrete-time i. $A(z)$, $B(z)$ and $C(z)$ are polynomials in the forward shift operator z. Despite the fact that $C(z)$ appears in the system equation, it will be treated as a design parameter: this can clearly be done by redefining ζ_i appropriately. In particular $C(z)$ is chosen such that $n_C = n_A - 1$. The choice of $C(z)$ is closely related to the choice of an observer polynomial

The *relative degree* ρ of the transfer function $\frac{\tilde{B}(z^{-1})}{\tilde{A}(z^{-1})}$ is equivalent to the system *discrete*

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time-delay

$k = \rho$. This can most readily be seen by rewriting the system as:

$$\tilde{A}(z^{-1})y_i = q^{-d}\tilde{B}(z^{-1})u_i + \tilde{C}(z^{-1})\zeta_i \tag{2}$$

where:

$$\tilde{A}(z^{-1}) = a_0 + a_1 z^{-1} + \cdots + a_n z^{-n_A} \tag{3}$$
$$\tilde{B}(z^{-1}) = b_0 + b_1 z^{-1} + \cdots + b_n z^{-n_B} \tag{4}$$

$$\tilde{C}(z^{-1}) = c_0 + c_1 z^{-1} + \cdots + c_n z^{-n_C} \tag{5}$$

For the purpose of prediction, it is convenient to rewrite Eq. (2) as:

$$y_{i+d} = \frac{\tilde{B}(z^{-1})}{\tilde{A}(z^{-1})}u_i + \frac{\tilde{C}(z^{-1})}{\tilde{A}(z^{-1})}\zeta_{i+d} \tag{6}$$

In this form, the *future* system output (that is the output d sample intervals into the future and assuming that i denotes the present time) is the sum of two terms:

1. $\frac{\tilde{B}(z^{-1})}{\tilde{A}(z^{-1})}u_i$ representing the effect of the *current* (u_i) and *past* (u_j; $j < i$) input signals and
2. $\frac{\tilde{C}(z^{-1})}{\tilde{A}(z^{-1})}\zeta_{i+d}$ representing *future* (ζ_j; $j > i$), *current* (ζ_i) and *past* (ζ_j; $j < i$) disturbances.

This decomposition of the disturbance can be made explicit by writing $\frac{\tilde{C}(z^{-1})}{\tilde{A}(z^{-1})}$ as:

$$\frac{\tilde{C}(z^{-1})}{\tilde{A}(z^{-1})} = \tilde{E}(z^{-1}) + q^{-d}\frac{\tilde{F}(z^{-1})}{\tilde{A}(z^{-1})} \tag{7}$$

where the degree of $\tilde{F}(z^{-1})$ is less than that of $\tilde{A}(z^{-1})$. *Algebraically* this is polynomial long division; *physically*, $q^{-d}\tilde{E}(z^{-1})\zeta_i$ represents the k *future* values of ζ_i and $\frac{\tilde{F}(z^{-1})}{\tilde{A}(z^{-1})}\zeta_i$ the current and past values.

Assuming that future values of ζ_i are unknown, but the best guess for them is zero, a *prediction* $\hat{y}_{i+d}$ for y_{i+d} at time i is obtained by deleting the first term of Eq. (7) and substituting into Eq. (6) to give:

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$$\hat{y}_{i+d} = \frac{\tilde{B}(z^{-1})}{\tilde{A}(z^{-1})} u_i + \frac{\tilde{F}(z^{-1})}{\tilde{A}(z^{-1})} \zeta_i \tag{8}$$

Rearranging Eq. (2) to give an expression for ζ_i and substituting into Eq. (8) gives the *predictor* equation:

$$\hat{y}_{i+d} = \frac{\tilde{G}(z^{-1})}{\tilde{A}(z^{-1})} u_i + \frac{\tilde{F}(z^{-1})}{\tilde{C}(z^{-1})} y_i \tag{9}$$

where

$$\tilde{G}(z^{-1}) = \tilde{E}(z^{-1})\tilde{B}(z^{-1}) \tag{10}$$

2.1.1. Initial conditions

Eq. (2) implicitly assumes that the system initial conditions are zero. If they are not, a term $\frac{\tilde{D}(z^{-1})}{\tilde{A}(z^{-1})}$ must be added to the right-hand side of Eq. (2) and $\frac{\tilde{E}(z^{-1})\tilde{D}(z^{-1})}{\tilde{C}(z^{-1})}$ to the right-hand side of Eq. (9).

However, if $\tilde{C}(z^{-1})$ has stable roots, then the predictor is stable and the effect of initial conditions has no long-term effect.

2.1.2. Stochastic Interpretation

If the disturbance signal ζ_i is (discrete-time) white noise (see *Models of Stochastic Systems*), then Eq. (9) gives the best predictor in the sense that it gives the least possible mean-square error. In essence, this is because the error

$$y_i - \hat{y}_i = z^k \tilde{E}(z^{-1})\zeta_i \tag{11}$$

represents the weighted sum of *future* values of white noise and is therefore uncorrelated (in the stochastic sense) with current and past measurements.

In this case, $\tilde{C}(z^{-1})$ is no longer a design parameter, but rather chosen based on noise model.

2.1.3. Generalized prediction

Two generalizations of the predictor of Section 2.1 are useful for control purposes (see Section **Error! Reference source not found.**).

Defining the *auxiliary output*

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$$\phi_i = \tilde{P}(z^{-1}) y_i \tag{12}$$

a predictor $\hat{\phi}_i$ for ϕ_i is:

$$\hat{\phi}_{i+d} = \frac{\tilde{G}(z^{-1})}{\tilde{C}(z^{-1})} u_i + \frac{\tilde{F}(z^{-1})}{\tilde{C}(z^{-1})} y_i \tag{13}$$

where

$$\tilde{G}(z^{-1}) = \tilde{E}(z^{-1})\tilde{B}(z^{-1}) \tag{14}$$

and

$$\frac{\tilde{P}(z^{-1})\tilde{C}(z^{-1})}{\tilde{A}(z^{-1})} = \tilde{E}(z^{-1}) + q^{-d} \frac{\tilde{F}(z^{-1})}{\tilde{A}(z^{-1})} \tag{15}$$

Notice that the generalized predictor of Eq. (13) reduces to the predictor of Eq. (9) when $\tilde{P}(z^{-1})$=1.

The interpretation of $\tilde{P}(z^{-1})$ is given in Section 3.

Using polynomials in the backward shift operator z^{-1} is useful in the preceding derivations as it has a clear physical interpretation. However, to link with the continuous-time approach it is useful to re-express Eqs. (13)-(15) in terms of *z*. Using Eqs. (3)-(5) and equivalent expressions for the other polynomials the predictor equations can be re-expressed as:

$$\hat{\phi}_{i+d} = \frac{G(z)}{C(z)} u_i + \frac{F(z)}{C(z)} y_i \tag{16}$$

$$G(z) = E(z)B(z) \tag{17}$$

$$z^{(k-1)} \frac{P(z)C(z)}{A(z)} = E(z) + \frac{F(z)}{A(z)} \tag{18}$$

2.2. Continuous-time model

Consider the continuous-time system (see *General Models of Dynamic Systems*) in transfer function form

$$A(s)y(t) = e^{-sT} B(s)u(t) + C(s)\zeta(t) \tag{19}$$

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where $y(t)$, $u(t)$, and $\zeta(t)$ are the system output, input and disturbance process at the continuous-time t. $A(s)$, $B(s)$ and $C(s)$ are polynomials in the Laplace operator s. The factor e^{-sT} represents a system *time-delay* of T units.

In a similar fashion to Eq. (12) define the *auxiliary output*

$$\phi(t) = e^{sT} P(s) y(t) \tag{20}$$

Combining Eqs. (19) and (20):

$$\phi(t) \frac{P(s)B(s)}{A(s)} u(t) + e^{sT} \frac{P(s)C(s)}{A(s)} \zeta(t) \tag{21}$$

In a similar manner to the discrete-time case, the transfer function multiplying $\zeta(t)$ has to be decomposed into *realizable* and *unrealizable* parts. This is considered in two stages:

1. unrealizable terms due to impulses
2. unrealizable terms due to non-causal terms

Considering first, the case where the delay $T = 0$ the relevant transfer function is decomposed as:

$$\frac{P(s)C(s)}{A(s)} = E_1(s) \frac{F_1(s)}{A(s)} \tag{22}$$

where the degrees of the polynomial are related by:

$$n_F = n_A - 1 \tag{23}$$

$$n_E = n_P + n_C - n_A \tag{24}$$

This is simply polynomial long division.

Secondly, consider the general case where the delay $T \geq 0$. Performing the decomposition of Eq. (22), the transfer function multiplying $\zeta(t)$ in Eq. (21) can be written as:

$$e^{sT} \frac{P(s)C(s)}{A(s)} = e^{sT} E_1(s) + e^{sT} \frac{F_1(s)}{A(s)} \tag{25}$$

The second term is no longer causal due to the factor e^{sT}. A further decomposition is then made:

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$$e^{sT}\frac{F_1(s)}{A(s)} = e^{sT}E_2(s) + \frac{F_2(s)}{A(s)} \tag{26}$$

where the *Finite-impulse response transfer function* $E_2(s)$ is anti-causal but $\frac{F_2(s)}{A(s)}$ is causal.

Combining the decompositions of Eqs. (22)-(26):

$$e^{sT}\frac{P(s)C(s)}{A(s)} = e^{sT}E(s) + \frac{F(s)}{A(s)} \tag{27}$$

where

$$E(s) = e^{sT}E_1(s) + E_2(s) \tag{28}$$
$$F(s) = F_2(s) \tag{29}$$

The "prediction" of $\phi(t)$ is then given as the realizable part of $\phi(t)$

$$\hat{\phi}(t) = \frac{P(s)B(s)}{A(s)}u(t) + \frac{F(s)}{A(s)}\zeta(t) \tag{30}$$

Eliminating $\zeta(t)$ using Eq. (21) and rearranging finally gives:

$$\hat{\phi}(t) = \frac{E(s)B(s)}{C(s)}u(t) + \frac{F(s)}{C(s)}y(t) \tag{31}$$

Example 1: Predictor

Consider the following first-order system:

$$A(s) = s \tag{32}$$

$$B(s) = b \tag{33}$$

The design polynomials are

$$P(s) = 1 + ps \tag{34}$$

$$C(s) = 1 \tag{35}$$

The decomposition of Eq. (22) becomes

$$\frac{1+ps}{s} = p + \frac{1}{s} \tag{36}$$

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Thus $E_1(s) = p$ and $F1(s) = 1$.

Assuming a unit time delay, the decomposition of Eq. (26)gives

$$F_2(s) = 1 \tag{37}$$

$$E_2(s) = \frac{1-e^{-s}}{s} \tag{38}$$

$E_2(s)$ of Eq. (38) is a FIR transfer function; the pole at $s = 0$ has zero residue. It can be implemented as written, but this leaves a canceling pole and zero at $s = 0$, it is better to approximate e^{-s} by, for example a Padé approximation and explicitly cancel.

Hence

$$\hat{\phi}(=)\left[p + \frac{1-e^{-s}}{s} \right] bu(t) + y(t) \tag{39}$$

2.3. Long-Range Prediction

The predictors defined in the preceding sections do not predict beyond a time corresponding to the system time-delay. It is, in fact, possible to predict further into the future *if some assumption is made about the future control signal*. Assuming that there is going to be no feedback in the future, (open-loop control) implies a deterministic control signal, which then allows such *long-range* prediction. This idea is used by Model-based Predictive Controllers (see *Model-based Predictive Controllers for Linear Systems*).

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Biographical Sketch

Peter J. Gawthrop was born in Seascale, Cumberland, in 1952. He obtained his BA (first class honors), MA and D.Phil. degrees in Engineering Science from Oxford University in 1973, 1977, and 1979, respectively. Following a period as a Research Assistant with the Department of Engineering Science at Oxford University, he became W.W. Spooner Research Fellow at New College, Oxford. He then moved to the University of Sussex as a Lecturer, and later a Reader in control engineering. Since 1987, he has held the Wylie Chair of Control Engineering in the Department of Mechanical Engineering at Glasgow University. He was involved in founding the Centre for Systems and Control, a cross-departmental research grouping at Glasgow with about twelve full time academic staff including four professors. His research interests include self-tuning control, continuous-time system identification and system modeling, particularly using bond graphs in the context of partially-known systems. He is interested in applying control techniques to a number of areas, including process control, robotics, aerospace systems and anesthesia. He has co-authored and authored some 120 conference and journal articles and three books in these areas. He was an associate editor of *Automatica* and an honorary editor of *IEE Proceedings Pt. D*, and serves on the editorial boards of a number of journals including the *IMechE Journal of Systems and Control*, *Journal of Process Control*, *IMA Journal of Mathematical Control and Information* and the *International Journal of Adaptive Control and Signal Processing* and the *European Journal of Control* In 1994 he was awarded the Honeywell International Medal by the Institute of Measurement and Control. In 1999 he spent a year in Australia at the Universities of Newcastle and Sydney.

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