

## TIME DELAY SYSTEMS

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**Keywords:** controllability, delay systems,  $\delta$ -freeness, distributed delays, infinite dimensional systems, quasi-finite systems, spectral controllability, tracking.

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### Summary

The presence of non-negligible time delays in a system leads to more complex analysis than in the finite dimensional case (i.e., without delay). This is also true for the controllability properties, central to automatic control. Indeed, this ubiquitous concept splits into many different notions when extended to the case of systems with delays. Instead of trying an impossible comparison between these notions, few notions which showed to be particularly useful in practice are detailed. Essential here is the use of feedforward methods to achieve tracking with stability. Two examples are treated in some detail, a first order system with delayed input and a delay system model for the torsional vibrations of a flexible rod modeled by an undamped wave equation.

## 1. Introduction

The occurrence of delays in engineering systems is manifold. Transportation of matter or information and propagation of electrical signals on transmission lines all generate delays. When the amplitudes of these delays cannot be neglected, compared to the process “response time”, a delay system model should be used and controllers should be designed based on this infinite dimensional model.

A very simple, though often useful, example of a delay system is

$$\dot{y}(t) = ay(t) + bu(t - h) + \varpi \quad (1)$$

where  $y(t)$  is the measured output,  $u(t)$  is the control input,  $h > 0$  is the amplitude of the delay, and  $\varpi$  is a constant disturbance. If  $a > 0$ , the system is unstable. This model is quite frequently used in industry, in particular if only poor knowledge is available from impracticable physical modeling. The multi variable generalization of this simple model yields a class of delay systems which are relatively easy to control (in the sense that the methods used are quite similar to those for systems without delays), yet arise quite frequently in practice. They are, therefore, called *quasi-finite* systems.

More complex models involve delays on the state (or the output) as well, and possibly on the derivative of the state. To deal with these, a more general structural property will be introduced (bearing a strong similarity with differential flatness of nonlinear systems): the  $\delta$ -freeness. As differential flat systems,  $\delta$ -free systems admit elegant solutions of tracking problems involving two main steps. First, open loop control laws are obtained from the reference trajectories without integrating any differential equation. Then, corrective laws are added in order to stabilize the evolution in a vicinity of these reference trajectories.

For simplicity of exposition, only single delay systems will be considered here (i.e., all delays are multiples of a single one), although the corresponding tools can equally well be used to treat multiple delays (with several “incommensurate” delays, that are linearly independent over the field  $\mathcal{Q}$  of rational numbers). Moreover, due to lack of space, only linear systems are considered.

The structure of the article is as follows. The second Section provides some insight in the occurrence of delays in distributed parameter models. The third Section is devoted to controllability notions, where it is seen that the picture is more complicated than for systems without delays. Several approaches, based on different mathematical tools, are considered to this end. Quasi-finite systems are treated in Section four. Their controllability properties are discussed. Moreover, stabilization and tracking methods are exposed and illustrated by a simulation example. Finally, a flexible rod example based on the undamped wave equation is treated in Section five. This example illustrates how delay system models can be derived from partial differential equations, how this can lead to systems involving delayed state derivatives, and how  $\delta$ -freeness can be introduced and exploited for the control of these more general distributed parameter systems.

## 2. Examples of Delay Systems Derived from Distributed Parameter Systems

A delay is naturally introduced in a model when there is a time lag between the control action and its influence on the system. Similarly, delays occur in an obvious manner if non-negligible measurement processing time is taken into account. In these cases, one readily obtains models of the quasi-finite type introduced in section four.

Delay system models can also be derived from some distributed parameter systems. Typically, such situations occur through the “transport equation”

$$\frac{\partial q}{\partial t} + v \frac{\partial q}{\partial z} = 0, \quad z \in (0, l), t > 0 \quad (2)$$

where  $q(z, t)$  may play the role of a reactant concentration or the temperature in a tube with plug flow (without mixing) in direction of increasing  $z$ . The parameter  $v > 0$  then represents the velocity of the flow, supposed to be constant. Similarly,  $q$  may stand for the thickness of a steel strip in a rolling mill transported with velocity  $v$  or the height of matter transported with a conveyer belt with speed  $v$ .

Often one may act on the system at  $z = 0$ , for example, by controlling the amount of material put on the belt at this place or the inflow concentration or temperature of the tube. This leads to a boundary condition of the type  $q(0, t) = u(t)$ . Suppose the quantity of interest is  $y(t) = q(l, t)$ . Then the value  $y(t)$  is given by  $y(t) = q(0, t - l/v)$ , which means one obtains the (elementary) delay system of the form

$$y(t) = u(t - l/v), \quad t \geq 0 \quad (3)$$

which just realizes a delay of amplitude  $l/v$ . The solutions of (2) are constant along the lines  $z - vt = \text{const.}$  in the  $z-t$ -plane, and can, thus, be written

$$q(z, t) = q_0(z - vt)h(z - vt) + u(t - z/v)h(vt - z)$$

where  $q(z, 0) = q_0(z)$ ,  $z \in [0, l]$  is the initial condition of (2) and  $h$  is the Heaviside function (the value of which is 0 for  $t < 0$  and 1 for  $t \geq 0$ ). It follows that  $u(t) = q_0(-vt)$ ,  $t \in [-l/v, 0]$  is the corresponding initial condition of the delay system (3). Both models have functional initial conditions (consisting of an infinite number of values). This is why they are called *infinite dimensional* systems.

This kind of transport phenomena is at the basis of many occurrences of delays in technological processes. For example, the tube considered may provide the inflow of an isothermal chemical stirred tank reactor with constant reaction volume in which a reactant is consumed through a first order chemical reaction. Then the concentration  $x$  of the reactant satisfies

$$\dot{x}(t) = a [u(t - l/v) - x(t)] - kx(t) + \varpi \quad (4)$$

where  $u(t)$  is the feed concentration at the input of the tube, while  $y(t) = u(t - l/v)$  is the concentration at the inflow of the reactor. The parameter  $a$  is proportional to the flow velocity  $v$  in the tube and  $k$  describes the reaction rate. Equation (4) is of the form of (1) with delay  $\tau = l/v$ . The perturbation  $\varpi$  models additive uncertainty.

As has been seen, the transport equation introduces a delay in an elementary fashion. Another class of delay phenomena is described by the undamped wave equation. It gives rise to so-called neutral delay systems, which are more difficult to control. An example of this type is an elastic rod undergoing torsional deformations, like a long motor shaft with a heavy tip load. This is treated in Section 5.

### 3. Controllability Notions for Linear Delay Systems

Here is a brief recall of controllability of finite dimensional systems, which admits various equivalent characterizations. Consider a single-input, time invariant finite dimensional linear system in state space form

$$\dot{x}(t) = Ax(t) + bu(t)$$

with  $x(t) = (x_1(t), \dots, x_n(t))$  the state,  $u(t)$  the input,  $A \in R^{n \times n}$ ,  $b \in R^{n \times 1}$ . Its controllability can, amongst others, equivalently be described by

Kalman's criterion:

$$rk_R(b, Ab, \dots, A^{n-1}b) = n,$$

the Hautus-Popov-Belevitch criterion:  $\forall s \in C, \quad rk_C[sl - A | b] = n,$

or the existence of a controller canonical form:

$$\dot{z}_1 = z_2, \dots, \dot{z}_{n-1} = z_n, \dot{z}_n = \alpha_1 z_1 + \dots + \alpha_n z_n + u$$

The generalizations to delay systems of the three characterizations of controllability recalled above do no longer coincide. Some of the many possible generalizations will be discussed now.

#### 3.1 Various approaches

Many different approaches for the study of time delay systems have been developed. In order to keep the exposition simple and brief, we only quote those most widely used.

- In the so-called systems over rings approach a system is viewed as a quadruple  $(A, B, C, D)$  of matrices. In the finite-dimensional case, the entries of those matrices are real numbers. In the case of delay systems, they stem from the ring  $R[\delta]$  of polynomials in the shift operator  $\delta$  (also called delay operator) with real coefficients. The delay operator  $\delta$  of amplitude  $h$  is defined by  $(\delta x_i)(t) = x_i(t - h)$ . The quadruple of matrices is then denoted as  $(A(\delta), B(\delta), C(\delta), D(\delta))$ , and the typical state-space equation reads

$$\dot{x} = A(\delta)x + B(\delta)u$$

$$y = C(\delta)x + D(\delta)u$$

where  $A(\delta) = A_0 + A_1\delta + \dots + A_\nu\delta^\nu$ ,  $B(\delta) = B_0 + B_1\delta + \dots + B_\mu\delta^\mu$ ,  $C(\delta) = C_0 + C_1\delta + \dots + C_{\bar{\nu}}\delta^{\bar{\nu}}$ ,

$D(\delta) = D_0 + D_1\delta + \dots + D_{\bar{\mu}}\delta^{\bar{\mu}}$ , which is the same as

$$\begin{aligned} \dot{x}(t) = & A_0x(t) + A_1x(t-h) + \dots + A_\nu x(t-\nu h) \\ & + B_0u(t) + B_1u(t-h) + \dots + B_\mu u(t-\mu h) \end{aligned} \quad (5)$$

$$\begin{aligned} y(t) = & C_0x(t) + C_1x(t-h) + \\ & \dots + C_{\bar{\nu}}x(t-\bar{\nu}h) + D_0u(t) + D_1u(t-h) + \dots + D_{\bar{\mu}}u(t-\bar{\mu}h) \end{aligned} \quad (6)$$

with state  $x(t) = (x_1(t), \dots, x_n(t))$ , input  $u(t) = (u_1(t), \dots, u_m(t))$ ,  $A_i \in R^{n \times n}$ ,  $i = 0, \dots, \nu$ ,  $B_j \in R^{n \times m}$ ,  $j = 0, \dots, \mu$ ,  $C_k \in R^{p \times n}$ ,  $k = 0, \dots, \bar{\nu}$ ,  $D_l \in R^{p \times m}$ ,  $l = 0, \dots, \bar{\mu}$ .

Several controllability properties have been extended to the systems over rings, such as Kalman's criterion (giving rise to the so-called "weak controllability") and Hautus-Popov-Belevitch's one (giving rise to the so-called "strong controllability"). These notions are defined mainly by generalizing the matrix criteria known from the finite dimensional case, as will be detailed in sections 3.2 to 3.5.

- In the functional analytic approach the values of the state belong to a function space, instead of  $R$  as in the finite dimensional (delay free) case.

More precisely, considering a system with delays:

$$\dot{x}(t) = \sum_{i=0}^{\nu} A_i x(t-ih) + B_0 u(t) \quad (7)$$

where  $h \in R$ ,  $x(t) \in R^n$ ,  $u(t) \in R^m$ ,  $A_0, \dots, A_\nu \in R^{n \times n}$ ,  $B_0 \in R^{n \times m}$ , the state of the system is taken as

$$z(t) = (x(t), x_t(\cdot))$$

where  $\forall t \geq 0$ ,  $x_t(\theta) = x(t + \theta)$  for  $\theta \in [-h, 0]$ , and  $x_t$  is a square integrable function ( $x_t \in L^2([-h, 0], R^n)$ ).

The operator  $e^{At}$  of systems without delays (yielding the solution of  $\dot{x} = Ax + Bu$ ) is generalized to a so-called "strongly continuous semigroup"  $S(t)$ , being an operator mapping an initial condition to a particular state. The evolution of the system is then described through the variation of constants formula

$$z(t) = S(t)\zeta + \int_0^t S(t-\tau)B_0u(\tau)d\tau \quad \text{for } t \geq 0$$

where the initial condition  $\zeta$  is defined by

$$x(0) = \zeta^0 \in R^n \quad \text{and} \quad \forall \theta \in [-h, 0] \quad x_0(\theta) = \zeta^1(\theta), \quad \text{where } \zeta^1 \in L^2([-h, 0], R^n).$$

Let  $\tilde{A}$  be the “infinitesimal generator” of  $S(t)$ : This is the generalization of the matrix  $A$  for the semigroup  $e^{At}$  of a finite dimensional system. It is defined as the limit

$$\tilde{A}z = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} (S(\varepsilon)z - z).$$

with appropriate domain (such that the limit exists).

The reachability subspace is the subspace of the state space which is reachable by means of a square integrable control.

Various controllability notions have been defined in this setting, namely the spectral controllability (discussed below), the exact controllability, where the reachability subspace is the whole state space, and the approximate controllability, where the reachability subspace is dense in the state space. With these notions, a system is considered to be controllable if there exists a control allowing one to steer the system from any one state to any other state. The difference with finite dimensional systems, thus, resides in the definition of the state space.

- In the behavioral setting, a system is considered to be given by a set of trajectories, called a behavior. Often, the considered trajectories are composed of  $C^\infty$  functions. The (behavioral) controllability is then defined as the possibility of concatenating any two system trajectories (see section 3.3).

### 3.6. Weak Controllability

This notion is one of the weakest controllability notions for delay systems. If a system is not weakly controllable, there is a severe obstruction to control, which is related to the existence of an autonomous subsystem. Therefore, this concept is useful mainly for characterizing non-controllability.

The definition of weak controllability is obtained by generalizing Kalman’s criterion in the system over rings approach, as follows. Equation (5) is called *weakly controllable* if

$$\text{rk}_{R[\delta]}(B(\delta), A(\delta)B(\delta), \dots, A(\delta)^{n-1}B(\delta)) = n$$

where  $R[\delta]$  is the ring of polynomials in the operator  $\delta$ , and  $\text{rk}_{R[\delta]}$  means that the rank is

determined through linear combinations with coefficients in  $R[\delta]$ . This notion corresponds to the absence of any “fixed trajectory” of the system (i.e., of a solution of an autonomous equation) as is illustrated on the following example.

### Example 3.1

The system  $\dot{x}(t) = x_1(t-h)$ ,  $\dot{x}_2(t) = x_2(t) + u(t-h)$  does not satisfy the rank condition,

since  $rk_R(B(\delta), A(\delta)B(\delta)) = rk_R \begin{pmatrix} 0 & 0 \\ \delta & \delta \end{pmatrix} < 2$ . There is a “fixed trajectory”, because  $x_1$  is the solution of the autonomous equation  $\dot{x}_1(t) = x_1(t-h)$  which cannot be influenced by the control  $u$ .

### Example 3.2

Take a simple example of the linearized equations for a satellite on a circular equatorial orbit

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\Omega^2 & 0 & 0 & 2\Omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\Omega & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t-h)$$

with  $x = (x_1, x_2, x_3, x_4) = (r, \dot{r}, \theta, \dot{\theta})$  and  $u = (u_1, u_2)$ . Here  $r$  and  $\theta$  designate the polar coordinates of the satellite in the equatorial plane,  $\Omega$  is the angular velocity, supposed constant,  $u_1$  represents radial thrust,  $u_2$  tangential thrust, and  $h$  is the communication delay (supposed constant) between the earth and the satellite.

Suppose now that the tangential thrust is not working, so that the system equations read:

$$\dot{x}_1(t) = x_2(t) \tag{8a}$$

$$\dot{x}_2(t) = 3\Omega^2 x_1(t) + 2\Omega x_4(t) + u_1(t-h) \tag{8b}$$

$$\dot{x}_3(t) = x_4(t) \tag{8c}$$

$$\dot{x}_4(t) = -2\Omega x_2(t) \tag{8d}$$

Subtracting  $\Omega x_4(t)/2$  on each side of Eq.(8b) yields

$$\dot{x}_2(t) - \frac{1}{2}\Omega x_4(t) = 3\Omega^2 x_1(t) + \frac{3}{2}\Omega x_4(t) + u_1(t-h)$$

Now, introducing  $\xi = \dot{x}_2(t) - (1/2)\Omega x_4(t) - u_1(t-h) = (3/2\Omega)(2\Omega x_1(t) + x_4(t))$  and differentiating leads to

$$\dot{\xi}(t) = 3\Omega^2 x_2(t) + \frac{3}{2}\Omega(-2\Omega)x_2(t) = 0$$

The quantity  $2\Omega x_1(t) + x_4(t) = 2\Omega r(t) + \theta(t)$  remains constant whatever the radial thrust may be. The physical interpretation of this invariant, or “fixed trajectory”, is that the radial thrust cannot modify the angular momentum.

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**Dr. Mounier** obtained his Doctoral degree in automatic control in 1995 from the Université Paris Sud, Orsay, France, when he was with the Laboratoire des Signaux et Systèmes, C.N.R.S. From 1998 to 2002 he was with the Institut d'Electronique Fondamentale at the same University. He is currently an assistant professor at the Robotics Center of the Ecole des Mines de Paris. His research interests include delay and communication systems, robotics with automotive applications and systems modeled by partial differential equations.

**Dr. Rudolph** received an engineering diploma in technical cybernetics from the Universität Stuttgart, Germany in 1989 and a "Doctorat" from the Université Paris XI, Orsay, France in 1991. He has been a fellow in the postdoctoral program and the "Habilitationenprogramm" of the Deutsche Forschungsgemeinschaft and an invited researcher at several research institutions in France. Currently he is a senior researcher ("Oberassistent") at the Institut für Regelungs-und Steuerungstheorie of the Technische Universität Dresden, Germany. His main research interests are in controller and observer design for nonlinear systems, algebraic systems theory, infinite dimensional systems, and applications.