GENERALISED MULTIDIMENSIONAL DISCRETE, CONTINUOUS-DISCRETE AND POSITIVE SYSTEMS

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Contents.

Summary

Different types of classical and generalised 2-D models and relationships between them are presented. Solutions to the models are given. The realisation problem for positive 2-D linear model are formulated and solved.

1. Introduction.

The most popular models of 2-D linear systems are models introduced by Roesser (1975), Fornasini and Marchesini (1976-1978) and Kurek (1985). Next the models have been extended for n-D (n>2) models with variable coefficients, singular 2-D models (Kaczorek 1988), singular 2-D continuos-discrete models and positive 2-D models.

The article is organised as follows. In Section 2 different types of classical and generalised 2-D and n-D models are presented. Relationships between the models and some transformations of 2-D models are discussed in Section 3. Solutions to the standard and generalised 2-D models are given in Section 4. Section 5 deals with singular 2-D continuos-discrete linear models. Positive 2-D linear models are considered in Section 6. Section 7 describes the realisation problem for positive 2-D linear systems.

2. Models of generalised multidimensional linear systems.

Roughly speaking, N-dimensional discrete systems are dynamical systems described by difference equations in N independent variables for $N \geq 2$.

Definition 1. A model

$$\begin{aligned} Ex_{i_1+1,i_2+1,\dots,i+1} = {} & A_0 x_{i_1,i_2,\dots,i_N} + \sum_{j=1}^{N} A_j x_{i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_N} \\ & + \sum_{1 \leq j<k \leq N} A_{jk} x_{i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_{k-1},i_k+1,i_{k+1},\dots,i_N+1} + \cdots \\ & + \sum_{j=1}^{N} A_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_N} x_{i_1+1,\dots,i_{j-1}+1,i_j,i_{j+1}+1,\dots,i_N+1} \\ & + B_0 u_{i_1,i_2,\dots,i_N} + \sum_{j=1}^{N} B_j u_{i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_N} \\ & + \sum_{1 \leq j<k \leq N} B_{jk} u_{i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_{k-1},i_k+1,i_{k+1},\dots,i_{N+1}+} + \cdots \\ & + \sum_{j=1}^{N} B_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_N} u_{i_1+1,\dots,i_{j-1}+1,i_j,i_{j+1}+1,\dots,i_N+1} \end{aligned} \tag{1a}$$

$$y_{i_1,i_2,\dots,i_N} = Cx_{i_1,i_2,\dots,i_N} + Du_{i_1,i_2,\dots,i_N} \quad i_1,i_2,\dots,i_N \in Z_+ = \{0,1,\dots\} \tag{1b}$$

$x_{i_1,i_2\dots i_N} \in R^n$ is the local semistate vector at the point $(i_1,i_2,\dots,i_N)$, $u_{i_1,i_2\dots i_N} \in R^m$ is the input, $y_{i_1,i_2\dots i_N} \in R^p$ is the output and $A_0, A_j, A_{jk}, \dots, A_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_N}$, $B_0, B_j, B_{jk}, \dots, B_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_N}$, C, D and E are real matrices of appropriate dimensions, is called the general N-dimensional model (GNDM).

Using the notations $I := (i_1,i_2,\dots,i_N)$, $i_k \in Z_+$, $k=1,\dots,N;$ $V := (1,1,\dots,1)$, e_j is the N-dimensional vector which is zero except in the jth entry where it is one, we may write (1) in the compact form

$$Ex(I+V)=A_0x(I)+\sum_{j=1}^{N}A_jx(I+e_j)+\cdots+\sum_{j=1}^{N}A_{1,\ldots,j-1,\;j+1,\ldots,N}x(I+V-e_j)+$$

$$+B_0u(I)+\sum_{j=1}^{N}B_ju(I+e_j)+\cdots+\sum_{j=1}^{N}B_{1,\ldots,j-1,j+1,\ldots,N}u(I+V-e_j) \qquad (1'a)$$

$$y(I)=Cx(I)+Du(I) \qquad (1'b)$$

where $x(I)=x_{i_1,i_2\ldots i_N}$, $u(I)=u_{i_1,i_2\ldots i_N}$, $y(I)=y_{i_1,i_2\ldots i_N}$.

The special feature of the generalised model is that the $q\times n$ E matrix may be singular. The model (1) is called singular if $q\neq n$ or $\det E=0$ when $q=n$.
Boundary conditions for (1a) are usually given by

$$x_{i_1,\ldots,i_{j-1},0,i_{j+1,\ldots,}i_N}=x_{j0} \qquad \textit{for} \qquad j=1,\ldots,N \qquad (2)$$

where x_{j0} are known vectors.
In the particular case for $N=2$, from (1) we obtain the generalised 2-D model (G2DM)

$$Ex_{i+1,j+1}=A_0x_{i,j}+A_1x_{i+1,j}+A_2x_{i,j+1}+B_0u_{i,j}+B_1u_{i+1,j}+B_2u_{i,j+1} \qquad (3a)$$
$$y_{i,j}=Cx_{i,j}+Du_{i,j} \qquad (3b)$$

From (3), for $B_1=B_2=0$, we obtain the first generalised Fornasini-Marchesini model (FGF-MM):

$$Ex_{i+1,j+1}=A_0x_{i,j}+A_1x_{i+1,j}+A_2x_{i,j+1}+B_0u_{i,j} \qquad (4a)$$
$$y_{i,j}=Cx_{i,j}+Du_{i,j} \qquad (4b)$$

From (3), for $A_0=0$ and $B_0=0$, we obtain the second generalised Fornasini-Marchesini model (SGF-MM):

$$Ex_{i+1,j+1}=A_1x_{i+1,j}+A_2x_{i,j+1}+B_1u_{i+1,j}+B_2u_{i,j+1} \qquad (5a)$$
$$y_{i,j}=Cx_{i,j}+Du_{i,j} \qquad (5b)$$

From (4), for $A_0=-A_1A_2$, we obtain the generalised Attasi model (GAM):

$$Ex_{i+1,j+1}=-A_1A_2x_{i,j}+A_1x_{i+1,j}+A_2x_{i,j+1}+B_0u_{i,j} \qquad (6a)$$
$$y_{i,j}=Cx_{i,j}+Du_{i,j} \qquad (6b)$$

Model (3),(4),(5) are called singular if matrix E is singular, i.e. $q \neq n$ or $\det E = 0$ when $q = n$.
Boundary conditions for (3a),(4a),(5a) and (6a) are given by

$$x_{i0} \text{ for } i = 0,1,... \text{ and } x_{0j} \text{ for } j = 0,1,... \tag{7}$$

or

$$x_{i,j} \text{ for all } i \text{ and } j \text{ such that } i + j = 0 \tag{8}$$

Definition 2. A model

$$Ex' = Ax + Bu \tag{9a}$$

$$y = Cx + Du \tag{9b}$$

where

$$x' = \begin{bmatrix} x^1_{i_1+1,i_2,...,i_N} \\ x^2_{i_1,i_2+1,...,i_N} \\ .... \\ x^N_{i_1,i_2,...,i_N+1} \end{bmatrix}, \quad x = \begin{bmatrix} x^1_{i_1,i_2,...,i_N} \\ x^2_{i_1,i_2,...,i_N} \\ .... \\ x^N_{i_1,i_2,...,i_N} \end{bmatrix}, \quad i_1, i_2, ..., i_N \in Z_+$$

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ .... \\ A_{N1} & \cdots & A_{NN} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix}, \quad C = [C_1, ..., C_N]$$

$x^j_{i_1,i_2,...,i_N} \in R^{n_j}$ is the *j*th local semistate vector for $j = 1,..., N$, $u = u_{i_1,i_2,...,i_N} \in R^m$ is the input, $y = u_{i_1,i_2,...,i_N} \in R^p$ is the output and A_{ij}, B_j, C_j, D and E are real matrices of appriopriate dimensions, is called the generalised N-dimensional Roesser model (GNDRM).
Using the compact notation we may write the model (9) in the form

$$Ex'(I) = Ax(I) + Bu(I) \tag{9'a}$$

$$y(I) + Cx(I) + Du(I) \tag{9'b}$$

where

$$x'(I) := \begin{bmatrix} x_1(I + e_1) \\ x_2(I + e_2) \\ \vdots \\ x_N(I + e_N) \end{bmatrix}$$

The special feature of the generalised model is that the $q \times n$ $(n = n_1 + ... + n_N)$ E matrix may be singular. The model (9) is called singular if $q \neq n$ or $\det E = 0$ when $q = n$. Boundary conditions for (9a) are usually given by (2).

In the particular case for $N = 2$ from (9) we obtain the generalised 2-D Roesser model (G2DRM)

$$E\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} + \begin{bmatrix} B_{10} \\ B_{20} \end{bmatrix} u_{i,j} \tag{10a}$$

$$y_{i,j} = \begin{bmatrix} C_{10} & C_{20} \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Du_{i,j} \tag{10b}$$

where $x^h_{i,j} \in R^{n_1}$, $x^v_{i,j} \in R^{n_2}$ are the horizontal and vertical local semistate vectors, respectively.

Definition 3. A model

$$\hat{E}x'(I) = \hat{A}x(I) + \hat{B}_0 u(I) + \hat{B}_1 u(I + e_1) + ... + \hat{B}_N u(I + e_N) \tag{11a}$$

$$y(I) = \hat{C}x(I) + \hat{D}u(I) \tag{11b}$$

where $x'(I)$, $X(I), u(I)$ and $y(I)$ are defined in the same way as for (9') is called the general N-D Rosser model with extended inputs (GNDREI). For $B_k = 0, \quad k = 1, ..., N$ we obtain the model (9').

Definition 4. A model

$$E\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} x^h_{i,j+1} \\ x^v_{i+1,j} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij} \tag{12a}$$

$$y_{ij} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Du_{ij} \qquad i, j \in Z_+ \tag{12b}$$

where $x^h_{ij} \in R^{n_1}$ and $x^v_{i,j} \in R^{n_2}$ are the horizontal and vertical semistate vectors, $u_{ij} \in R^m$ and $y_{ij} \in R^p$ are the input and output vectors, $A_{11}, F_1 \in R^{n_1 \times n_1}$, $A_{22}, F_2 \in R^{n_2 \times n_2}$, $E \in R^{n \times n}$, $n = n_1 + n_2$, $B_1 \in R^{n_1 \times m}$, $B_2 \in R^{n_2 \times m}$, $C_1 \in R^{p \times n_1}$, $C_2 \in R^{p \times n_2}$, $D \in R^{p \times m}$, is called the 2-D Roesser model with extended states (2DRMES).

The models (1)-(11) are called standard if matrix E is nonsingular. If E is nonsingular, then premultiplying (1a),(3a),(4a),(5a),(6a),(9a),(10a) and (11a) by E^{-1} we obtain corresponding standard models with $E = I$.

If the entries of the matrices depend on the variables $i_1, i_2, ..., i_N$ the models are called the models with variable coefficients.

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Biographical Sketch

Tadeusz Kaczorek received the MSc.; PhD and DSc degrees from Electrical Engineering of Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the period of 1968-69 he was the dean of Electrical Engineering Faculty an in the period 1970-73 he was the prorector of Warsaw University of Technology. Since 1971 he has been professor and since 1974 full professor at Warsaw University of Technology. In 1986 he was elected a member of Polish Academy of Sciences. in the period 1988-91 he was the director of the Research Centre of Polish Academy of Sciences in Rome. His research interests cover the theory of systems and the automatic control systems theory, specially, singular multidimensional systems, positive multidimensional systems and singular positive 1D and 2D systems. He has published 18 books (5 in English) and over 600 scientific papers in journals like IEEE Transactions on Automatic Control, IEEE Transactions on Neural Networks, Multidimensional Systems and Signal Processing, International Journal of Control etc. He has presented more than 80 invited papers on international conferences and world congresses. He has given invited lectures in more than 50 universities in USA, Canada, UK , German, Italy, France, Japan, Greece etc. He has been member of many international committees and editorial boards.