STABILITY OF 2D SYSTEMS

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Contents

Summary

This chapter deals with the formulation of stability theory for two-dimensional dynamical systems, by generalizing the concepts used in the classical one-dimensional case. Both the input-output description and different forms of state-space representation are considered, leading to the definition of bounded-input bounded-output stability and asymptotic stability, with a discussion of the relationship between them. Necessary and sufficient conditions for stability are given, using functions of two complex variables, and the Nyquist stability criterion for feedback systems is extended to the two-dimensional case. Most of the analysis refers to systems where both independent variables, corresponding to the two dimensions, are discrete, but the cases of discrete-continuous and doubly continuous systems are also addressed. Finally, some application areas are discussed, specifically in compartmental modeling and iterative learning control.

1. Introduction

By a two-dimensional (2D) dynamical system, we mean one which is described by a set of differential or difference equations in two independent variables. According to the particular application considered, both variables may be continuous, or both discrete, or one continuous and the other discrete. In any case, however, in order to introduce the concept of stability, we have to assume a sequential ordering for at least one independent variable, so as to make a distinction between past and future, by analogy with time in a one-dimensional (1D) system. We can then say, loosely speaking, that a system is stable if the effects of past disturbances or initial conditions die away as the system evolves into the future, and this notion can be given a satisfactory mathematical expression, at least in the case of systems described by linear equations with constant coefficients. For nonlinear systems or those with time-dependent parameters, on the other hand, the situation is much more complicated and less well understood. Moreover, even for the linear time-invariant case, different formulations of the stability concept can be given, which are not necessarily equivalent.

2. Discrete Systems

Just as a 1D system, a 2D system can be described by either an input-output relation or a set of state-space equations. The input-output description of a causal linear shift-invariant 2D system can be written

$$y(m,n)=\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} g_{hk}u(m-h,n-k)\,, \tag{1}$$

where u and y are the input and output respectively, and the coefficients g_{hk} specify the impulse response of the system. A state-space representation of such a system can be put in the form

$$\mathbf{x}(h+1,k+1)=\mathbf{A}_1\mathbf{x}(h,k+1)+\mathbf{A}_2\mathbf{x}(h+1,k)+\mathbf{B}_1u(h,k+1)+\mathbf{B}_2u(h+1,k) \tag{2}$$

$$y(h,k)=\mathbf{C}\mathbf{x}(h,k)+g_{00}u(h,k)\,, \tag{3}$$

where $\mathbf{x}$ is a state vector of sufficient length and the coefficient matrices are of compatible sizes. This representation, known as a Fornasini-Marchesini model, is naturally not unique, nor indeed is its structure, since the 2D environment admits considerably more variety of system description than is possible in the 1D case, but it is generally applicable and convenient. An alternative representation, with the structure

$$\begin{bmatrix}\mathbf{x}_1(h+1,k)\\ \mathbf{x}_2(h,k+1)\end{bmatrix}=\mathbf{A}\begin{bmatrix}\mathbf{x}_1(h,k)\\ \mathbf{x}_2(h,k)\end{bmatrix}+\mathbf{B}u(h,k) \tag{4}$$

$$y(h,k)=\mathbf{C}_1\mathbf{x}_1(h,k)+\mathbf{C}_2\mathbf{x}_2(h,k)+g_{00}u(h,k) \tag{5}$$

is known as a Roesser model. These two forms of representation are in fact equivalent, in the sense that either can be transformed into the other.

The transfer function of the system defined by Eq. (1) is

$$G(z_1, z_2) = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} g_{hk} z_1^h z_2^k \quad (6)$$

giving the relation

$$Y(z_1, z_2) = G(z_1, z_2) U(z_1, z_2) \quad (7)$$

between the 2D z-transforms of the input and output, where (z_1, z_2) correspond to *backward shift* operations, following the usual practice in 2D, though not 1D, discrete system theory. If the transfer function is expressible as a rational fraction

$$G(z_1, z_2) = \frac{\psi(z_1, z_2)}{\phi(z_1, z_2)} , \quad (8)$$

where ϕ and ψ are polynomials, then a state-space representation can be constructed, either in the form of Eqs. (2) and (3), or alternatively Eqs. (4) and (5). Conversely, an expression of the form (8) can, for instance, be obtained from the Fornasini-Marchesini equations by setting

$$\phi(z_1, z_2) = \det(\mathbf{I} - z_1\mathbf{A}_1 - z_2\mathbf{A}_2) \quad (9)$$

$$\psi(z_1, z_2) = \mathbf{C}\operatorname{adj}(\mathbf{I} - z_1\mathbf{A}_1 - z_2\mathbf{A}_2)(z_1\mathbf{B}_1 + z_2\mathbf{B}_2) + g_{00}\phi(z_1, z_2) \quad (10)$$

although it is not necessarily in its lowest terms, as the numerator and denominator polynomials may have common factors. Incidentally, in contrast to the 1D case, 2D polynomials can simultaneously vanish without having any factor in common, for example if $\phi(z_1, z_2) = 2 - z_1 - z_2$ and $\psi(z_1, z_2) = (1 - z_1)(1 - z_2)$, when $G(z_1, z_2)$ is said to have a *non-essential singularity of the second kind*.

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Bibliography

Cook P.A. (2000). Stability of two-dimensional feedback systems. *International Journal of Control* **73**(4), 343–348. [Extends the Nyquist stability criterion to cover 2D systems].

Fornasini E., Marchesini G. (1980). Stability analysis of 2d systems. *IEEE Transactions on Circuits and Systems* **27**(12), 1210–1217. [Develops asymptotic stability conditions for 2D systems in state-space form].

Goodman D. (1977). Some stability properties of two-dimensional linear shift-invariant digital filters. *IEEE Transactions on Circuits and Systems* **24**(4), 201–208.[Investigates the relationship between BIBO and other definitions of input-output stability for 2D systems].

Hmamed A. (1997). Component-wise stability of 1d and 2d linear discrete-time systems. *Automatica* **33**(9), 1759–1762. [Introduces component-wise stability for discrete systems].

Rogers E., Owens D.H. (1992). *Stability Analysis for Linear Repetitive Processes*. New York: Springer Verlag. [Gives a comprehensive treatment of stability properties for multi-pass processes].

Valcher M.E. (1997). On the internal stability and asymptotic behavior of 2d positive systems. *IEEE Transactions on Circuits and Systems – I* **44**(7), 602–613. [Derives conditions for stability of 2D state-space models in which all coefficients are non-negative].

Biographical Sketch

Peter A. Cook has been working in the area of Control and Systems Theory since 1971, and was awarded the degree of Doctor of Science by the University of Manchester in 1984 for his work on Analysis and Control of Dynamical Systems. He currently holds the position of Senior Lecturer in the Control Systems Centre, Department of Electrical Engineering & Electronics, at UMIST in Manchester, UK. His main research interests at present are in Nonlinear Systems, Adaptive and Intelligent Control, Two-Dimensional System Theory, and Dynamics of Vehicle Convoys. Dr. Cook is a Fellow of the Institute of Mathematics and its Applications.