

## INPUT–OUTPUT STABILITY

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**Keywords:** stability, input–output systems, gain, circle theorem, passivity

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### Summary

In this article we define input–output stability for a system by considering the relative “size” of signals at the input and output. This leads to the notion of gain, which generalizes the gain of a linear amplifier. The small-gain theorem is the basic result, and states that the product of the gains in a feedback loop should be less than one for stability. After considering linear systems, we discuss the case of bilinear systems, and obtain an expression for the gain of such systems. The circle theorem for feedback systems containing a linear element and a nonlinear element in the feedback loop is covered in detail, and generalizations based on passivity and multipliers are introduced. Finally the robustness of stability under parameter changes and input disturbances is discussed.

### 1. Introduction

The notion of stability is one of the most important concepts in the whole of systems theory. An overall closed-loop system must be stable even if the plant has been deliberately designed to be unstable (as in the case of modern fighter aircraft, for example). This intuitive idea of stability, that systems should not produce signals of unbounded growth for input signals that are bounded in some sense, can be formalized in a number of ways; the most popular and effective are the Lyapunov notion of stability based on internal energy dissipation, and the input–output definition based essentially on the “gain” of the system. Much of the input–output theory can be traced back to the work of Zames, Sandberg, and Popov. The gain of a linear amplifier is, of course, just the ratio of the output and the input voltages, which is a positive number. We shall see that this simple notion of gain generalizes to many kinds of nonlinear systems, provided we have a concept of “size” on the input and output signals, and we

shall do this by introducing a “norm” on these signals. In general, we shall see that the gain is then not necessarily constant, but can be a nonlinear positive real-valued function.

Once we have defined input–output stability for a general (open-loop) system, it will be clear that it is not equivalent to Lyapunov stability in general, although for controllable and observable linear systems the two concepts coincide. When we consider systems in the form of a feedback loop with a subsystem in the feedforward path and one in the feedback path, such as the one in Figure 1, then we will show that the most basic result is the celebrated “small-gain theorem,” which states that if the product of the gains of  $S_1$  and  $S_2$  is less than one, then the overall system is input–output stable. In many cases, one of the subsystems  $S_1, S_2$  is linear and we can obtain generalizations of the Nyquist stability criterion in the form of Popov’s theorem or the various circle criteria, which state that the frequency response of the linear part must lie in a certain circle whose radius and center are specified by the nonlinear part. Finally, most systems are not known completely: some parameters may be unknown, the model may be only approximate, and there may be unknown disturbances in the system. It is therefore necessary to know that the stability of some nominal system implies that of systems perturbed around the nominal one and also what “size” of perturbations can be allowed. This is the theory of robustness, and we shall mention the most basic results.

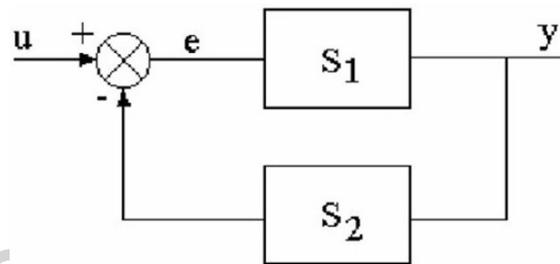


Figure 1. A simple feedback system

## 2. Signals and Norms

A signal in a system is taken as some vector-valued function  $t \rightarrow f(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  of time into Euclidean  $n$ -space (although it is sometimes useful to consider complex vector-valued functions). Such a signal may be an input or an output of a system. (Note, however, that for distributed parameter systems, described by partial differential or functional differential equations, the image space must be some infinite-dimensional vector space.) In order to define the stability of a system, we must have some notion of “size” of a signal in order to measure the relative size of the output and input signals to the system. This is usually done by means of a ‘norm’. First, on the image space  $\mathbb{R}^n$  of the signals we place some  $n$ -dimensional norm, such as

$$\|x\|_2 = \left( \sum_{i=1}^n x_i^2 \right)^{1/2} \quad \text{or} \quad \|x\|_p = \left( \sum_{i=1}^n x_i^p \right)^{1/p}, \quad p > 1$$

or

$$\|x\|_{\infty} = \max_i |x_i|.$$

The first is the most common (the standard Euclidean norm) and we usually omit the subscript “2.” Now, given a signal  $f(t)$  we may define its norm in many ways. Again, here are some of the most common (note that the signal may be very general: all that is really required is the formal mathematical condition of “measurability” so that the integrals are well defined):

$$\|f\|_2 = \left( \int_0^{\infty} \|f(t)\|^2 dt \right)^{1/2}$$

$$\text{or } \|f\|_p = \left( \int_0^{\infty} \|f(t)\|^p dt \right)^{1/p}, 1 \leq p < \infty$$

$$\text{or } \|f\|_{\infty} = \sup_{t \in [0, \infty)} \|f(t)\|$$

(where the norm  $\|f(t)\|$  of the function value is a suitable norm such as one of those above). Again, the first and last are the most commonly used; the first is essentially a measure of the total energy in the signal, while the last measures absolute size at each time. The first has the desirable property that (by Parseval’s theorem) it has the same value (modulo  $2\pi$ ) as that of the Fourier transform of the signal:

$$\int_0^{\infty} \|f(t)\|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|F(j\omega)\|^2 d\omega$$

(i.e. power can be measured in the “state space” or the frequency domain). We must also consider the spaces of all signals for which these norms are finite, so we introduce the spaces

$$L_p(0, \infty) = \{f : \|f\|_p < \infty\}, 1 \leq p \leq \infty.$$

However, since we want to allow signals for which these norms are not necessarily bounded, we also introduce more general spaces of functions  $f$ , whose “truncation”  $f_T$ , defined by

$$f_T(t) = \begin{cases} f(t) & 0 \leq t \leq T \\ 0 & t > T \end{cases}$$

belongs to the appropriate space, i.e.

$$L_p^e(0, \infty) = \{f : \|f_T\|_p < \infty, \text{ for all } T > 0\}.$$

These are the *extended*  $L_p$  spaces. Note that

$$L_p(0, \infty) \subseteq L_p^e(0, \infty) \text{ for all } p \geq 1.$$

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### Biographical Sketch

**S.P. Banks** gained his B.Sc. in 1960, M.Sc. in 1971, and Ph.D. in 1974, all in mathematics. After completing a senior research fellowship in the Control Theory Centre, University of Warwick, and a short stay in British Aerospace, he joined the Department of Control Engineering (now the Department of Automatic Control and Systems Engineering), University of Sheffield as a lecturer in 1980 and became a full professor in 1990. He has published over 130 papers and six books and is currently editor of *IMA Journal of Mathematical Control and Information* and an associate editor of *Computational and Applied Mathematics*.