DESIGN FOR NONLINEAR CONTROL SYSTEMS

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Contents

Summary

One of the basic fundamental issues in control theory is the ability to design a feedback law to the purpose of robustly stabilizing a system, in the presence of structured uncertainties, such as parameter variations as well as un-structured uncertainties, such as unmodeled dynamics. In the case of finite-dimensional, time-invariant, linear models, a variety of techniques are available, which range from elementary methods, based on Nyquist criterion and root locus analysis, to more elaborate methods, based on the design of feedback laws so as to keep the norm of certain operators below a fixed level. In recent years, some of these methods have been successfully extended to certain classes of nonlinear system. This chapter describes some of these techniques, with special emphasis on recursive methods for the synthesis of feedback laws which assign a fixed Lyapunov function or a fixed input-to-state-gain.

1. Introduction

One of the basic fundamental issues in control theory is the ability to design a feedback law for the purpose of robustly stabilizing a system, in the presence of structured uncertainties, such as parameter variations. In the case of finite-dimensional, time-invariant, linear models, a variety of techniques are available, which range from elementary methods based on the properties of the root locus (which lends itself to a very intuitive characterization of the "stability margin" inherent to certain design techniques, such as "small-gain" and/or "high-gain") to more sophisticated methods such as, in the case of parameter uncertainties, those relying upon the analysis of the closed-loop characteristic polynomial (whose uncertain coefficients range over prescribed intervals) or the design of a feedback controller imposing a fixed (parameter independent) Lyapunov function. In the recent years, similar methods have been gradually developed also for nonlinear systems. In this chapter, we have chosen to review in some detail a number of these methods. Of course, stabilization is just one aspect of feedback design, for linear as well for nonlinear systems. It must be stressed, though, that stability is the most important feature a closed-loop system is required to have. It is for this reason that understanding how a nonlinear system can be stabilized, and – above all – robustly stabilized, is perhaps the most important step needed to understand how a nonlinear system can be controlled. Unless otherwise specified, we deal in what follows with single-input single-output nonlinear systems modeled in state-space form by equations of the type

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

in which $f(x), g(x), h(x)$ are smooth functions of x.

2. State-feedback Design for Global Stability

We present in this section a method for global stabilization of systems described by equations having a *lower-triangular* structure, namely systems modeled by equations of the form

$$\begin{aligned} \dot{z} &= f(z,\xi_1) \\ \dot{\xi}_1 &= q_1(z,\xi_1) + b_1(z,\xi_1)\xi_2 \\ \dot{\xi}_2 &= q_2(z,\xi_1,\xi_2) + b_2(z,\xi_1,\xi_2)\xi_3 \\ &\cdots \\ \dot{\xi}_r &= q_r(z,\xi_1,\ldots,\xi_r) + b_r(z,\xi_1,\ldots,\xi_r)u \end{aligned} \tag{2}$$

in which $z \in \mathbb{R}^n, \xi_1 \in \mathbb{R}$, for $i = 1,\ldots,r, u \in \mathbb{R}$. It is assumed that $f(0,0) = 0$ and that

$$b_i(z,\xi_1,\ldots,\xi_i) \neq 0$$

for all $(z,\xi_1,\ldots,\xi_i) \in \mathbb{R}^n \times \mathbb{R}^i$ and all $i = 1,\ldots,r$.

The design of a feedback law that globally stabilizes this class of systems is based on a modular technique, currently known as "backstepping", which is based on a recursive use of the following pair of simple lemmas.

Lemma 1 *Consider a system described by equations of the form*

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= q(z,\xi) + b(z,\xi)u \end{aligned} \tag{3}$$

in which $(z,\xi) \in \mathbb{R}^n \times \mathbb{R}$, $f(0,0) = 0$ *and* $b(z,\xi) \neq 0$. *Suppose there exists a smooth real-valued function* $V(z)$, *which is positive definite and proper, such that*

$$\frac{\partial V}{\partial z} f(z,0) < 0$$

for all nonzero z. *Then, there exists a smooth state feedback law* $u - u(z,\xi)$ *and a smooth real-valued function* $W(z,\xi)$, *which is positive definite and proper, such that*

$$\begin{pmatrix} \dfrac{\partial W}{\partial z} & \dfrac{\partial W}{\partial \xi} \end{pmatrix} \begin{pmatrix} q(z,\xi) + \overset{f(z,\xi)}{b(z,\xi)} u(z,\xi) \end{pmatrix} < 0$$

for all nonzero (z,ξ).

In fact, observe that the function $f(z,\xi)$ can be put in the form

$$f(z,\xi) = f(z,0) + p(z,\xi)\xi \tag{4}$$

where $p(z,\xi)$ is a smooth function. Then, consider the positive definite and proper function

$$W(z,\xi) = V(z) + \frac{1}{2}\xi^2, \tag{5}$$

and observe that

$$\begin{pmatrix} \dfrac{\partial W}{\partial z} & \dfrac{\partial W}{\partial \xi} \end{pmatrix} \begin{pmatrix} q(z,\xi) + \overset{f(z,\xi)}{b(z,\xi)} u(z,\xi) \end{pmatrix} = \frac{\partial V}{\partial z} f(z,0) + \frac{\partial V}{\partial z} p(z,\xi)\xi + \xi(q(z,\xi) + b(z,\xi)u) \cdot$$

Choosing

$$u = u(z,\xi) = \frac{1}{b(z,\xi)}\left(-q(z,\xi) - \xi - \frac{\partial V}{\partial z} p(z,\xi) \right) \tag{6}$$

yields the required result.

In view of the classical (direct and converse) Lyapunov Theorems, the hypothesis of this Lemma (namely the hypothesis of the existence of a smooth positive definite and proper function $V(z)$ such that $\frac{\partial V}{\partial z} f(z,0)$ is negative for each nonzero z) is equivalent to the hypothesis that the equilibrium $z=0$ of the subsystem

$$\dot{z} = f(z,0)$$

is globally asymptotically stable. On the other hand, by the direct Lyapunov Theorem, the conclusion of the Lemma implies that the equilibrium at $(z,\xi)=(0,0)$ of the system

$$\dot{z} = f(z,\xi)$$

$$\dot{\xi} = q(z,\xi) + b(z,\xi)u(z,\xi)$$

is globally asymptotically stable. Thus, the result of the Lemma simply says that, if $\dot{z} = f(z,0)$ has a globally asymptotically stable equilibrium at $z=0$, then the equilibrium $(z,\xi)=(0,0)$ of system (3) can be globally asymptotically stabilized by means of a smooth feedback law $u = u(z,\xi)$.

The next Lemma (which contains Lemma 1 as a particular case) extends this result, by showing that, to the purpose of stabilizing the equilibrium $(z,\xi)=(0,0)$ of system (3), it suffices to assume that the equilibrium $z=0$ of $\dot{z} = f(z,\xi)$ is *stabilizable* by means of a smooth control law $\xi = v^{*}(z)$.

Lemma 2 *Consider a system described by equations of the form* (3), *in which* $(z,\xi)\in\mathbb{R}^n\times\mathbb{R}$, $f(0,0)=0$ *and* $b(z,\xi)\neq 0$. *Suppose there exists a smooth real-valued function*

$$\xi = v^{\star}(z),$$

with $v^{\star}(0)=0$, *and a smooth real-valued function* $V(z)$, *which is positive definite and proper, such that*

$$\frac{\partial V}{\partial z} f(z, v^{\star}(z)) < 0$$

for all nonzero z. *Then, there exists a smooth state feedback law* $u = u(z,\xi)$ *and a smooth real-valued function* $W(z,\xi)$, *which is positive definite and proper, such that*

$$\begin{pmatrix} \frac{\partial W}{\partial z} & \frac{\partial W}{\partial \xi} \end{pmatrix} \left(\overset{f(z,\xi)}{q(z,\xi) + b(z,\xi)u(z,\xi)} \right) < 0$$

for all nonzero (z,ξ).

In fact, it suffices to consider the globally defined change of variables

$$y = \xi - v^\star(z),$$

which transforms (3) into a system of the form

$$\begin{aligned} \dot{z} &= f(z, v^\star(z) + y) \\ \dot{y} &= -\frac{\partial v^\star}{\partial z} f(z, v^\star(z) + y) + q(z,\xi) + b(z,\xi)u, \end{aligned} \tag{7}$$

and to observe that the feedback law

$$u = \frac{1}{b(z,\xi)} \left(\frac{\partial v^\star}{\partial z} f(z, v^\star(z) + y \right) + u'$$

changes the latter into a system satisfying the hypotheses of Lemma 1.

Using repeatedly the property indicated in Lemma 2 it is straightforward to derive the following stabilization result about a system in the form (2)

Theorem 1 *Consider a system of the form* (2), *in which* $(z, \xi_1, ..., \xi_r) \in \mathbb{R}^n \times \mathbb{R}^r$ $f(0,0) = 0$

and $b_i(z, \xi_1, ..., \xi_i) \neq 0$

for all $(z, \xi_1, ..., \xi_i) \in \mathbb{R}^n \times \mathbb{R}^i$ *and all* $i = 1, ..., r$. *Suppose there exists a smooth real-valued function*

$$\xi_1 = v^\star(z),$$

with $v^\star(0) = 0$, *and a smooth real-valued function* $V(z)$, *which is positive definite and proper, such that*

$$\frac{\partial V}{\partial z} f_0(z, v^\star(z)) < 0$$

for all nonzero z. *Then, there exists a smooth state feedback law*

$$u = u(z, \xi_1, ..., \xi_r)$$

which globally asymptotically stabilizes the equilibrium

$$(z, \xi_1, ..., \xi_r) = (0, 0, ..., 0)$$

of the corresponding closed loop system.

Of course, a special case in which the result of Theorem 1 holds is when $v^*(z) = 0$, i.e., when $\dot{z} = f(z, 0)$ has a globally asymptotically stable equilibrium at $z = 0$. This important case occurs when a system form (2), with output

$$y = \xi_1$$

has a *zero dynamics* with a globally asymptotically stable equilibrium at $z = 0$. For convenience this special case in summarized in the Corollary of Theorem 1.

Corollary 1 *Consider a system of the form* (2), *with output* $y = \xi_1$. *Suppose its zero dynamics have a globally asymptotically stable equilibrium at* $z = 0$. *Then, there exists a smooth state feedback law*

$$u = u(z, \xi_1, ..., \xi_r),$$

which globally asymptotically stabilizes the equilibrium

$$(z, \xi_1, ..., \xi_r) = (0, 0, ..., 0)$$

of the corresponding closed loop system.

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Biographical Sketch

Alberto Isidori was born in Rapallo, Italy, in 1942. His research interests are primarily focused on mathematical control theory and control engineering. He graduated in electrical engineering from the University of Rome in 1965. Since 1975, he has been Professor of Automatic Control in this University. Since 1989, he is also affiliated with the Department of Systems Science and Mathematics at Washington University in St. Louis. He has held visiting positions at various academic/research institutions which include the University of Illinois at Urbana-Champaign, the University of California at Berkeley, the ETH in Zürich and the NASA-Langley research center.

He is the author of several books, including "Nonlinear Control Systems" (Springer Verlag), 1985, 1989 and 1995; "Nonlinear Control Systems II" (Springer Verlag), 1999. He is author of more than 80 articles on archival journals, of 16 book chapters and more than 100 papers on refereed conference proceedings, for a large part on the subject of nonlinear feedback design. He is also also editor/coeditor of 16 volumes of Conference proceedings.

He received the G.S. Axelby Outstanding Paper Award in 1981 and in 1990. He also received the Automatica Best Paper Award in 1991. In 1987 he was elected Fellow member of the IEEE "for fundamental contributions to nonlinear control theory". In 1996, Dr. Isidori received from IFAC the

"Georgio Quazza Medal" for "pioneering and fundamental contributions to the theory of nonlinear feedback control". In 2000 he was awarded the first ``Ktesibios Award" from the Mediterranean Control Association. In 2001, he received the ``Bode Lecture Prize" from the Control Systems Society of IEEE. From 1993 to 1996 he served in the Council of IFAC. From 1995 to 1997 he was President of the European Community Control Association. Dr. Isidori is listed in the Highly-Cited database (http://isihighlycited.com) among the top most-cited authors in Engineering.