FLATNESS BASED DESIGN

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Contents

Summary

Motion planning, i.e., steering a system from one state to another, is a basic question in automatic control. For a certain class of systems described by ordinary differential equations and called flat systems motion planning admits simple and explicit solutions. This stems from an explicit description of the trajectories by an arbitrary time function y, the flat output, and a finite number of its time derivatives. Such explicit descriptions are related to old problems on Monge equations and equivalence investigated by Hilbert and Cartan. A system equivalence relation, using the framework of differential geometry of jets and prolongations of infinite order, is sketched. In this setting, two systems are said to be equivalent if any variable of one system may be expressed as a function of the variables of the other system and of a finite number of their time derivatives. Equivalence is presented in an elementary way and illustrated on the VTOL example: it corresponds to *endogenous feedback* transformations, *i.e.*, a

special type of dynamic feedback. *Differentially flat* systems are then defined as systems equivalent to linear controllable ones. Consequently flat systems are linearizable by endogenous feedback. The endogenous linearizing feedback is explicitly computed in the case of the VTOL aircraft to track given reference trajectories with stability. Deciding whether a given system is flat or not, is an open problem. We give some partial results such as the ruled manifold criterion, a simple necessary condition that can be very useful to prove that a system is not flat.

1. Introduction

In this chapter we concentrate on a specific class of systems, called "(differentially) flat systems", for which the structure of the trajectories of the (nonlinear) dynamics can be completely characterized. Flat systems are a generalization of linear systems (in the sense that all linear, controllable systems are flat), but the techniques used for controlling flat systems are quite different from many of the existing techniques for linear systems. As we shall see, flatness is particularly well tuned for allowing one to solve the inverse dynamics problems and one builds off of that fundamental solution in using the structure of flatness to solve more general control problems.

Flatness was first defined by Fliess et al. using the formalism of differential algebra. In differential algebra, a system is viewed as a differential field generated by a set of variables (states and inputs). The system is said to be flat if one can find a set of variables, called the flat outputs, such that the system is (non-differentially) algebraic over the differential field generated by the set of flat outputs. Roughly speaking, a system is flat if we can find a set of outputs (equal in number to the number of inputs) such that all states and inputs can be determined from these outputs without integration. More precisely, if the system has states $x \in \mathbb{R}^n$, and inputs $u \in \mathbb{R}^m$ then the system is flat if we can find outputs $y \in \mathbb{R}^m$ of the form

$$y = h(x, u, \dot{u}, \ldots, u^{(r)})$$

such that

$$x = \varphi(y, \dot{y}, \ldots, y^{(q)})$$
$$u = \alpha(y, \dot{y}, \ldots, y^{(q)}).$$

Flatness has been defined in a more geometric context, where tools for nonlinear control are more commonly available. One approach is to use exterior differential systems and regard a nonlinear control system as a Pfaffian system in an appropriate space. In this context, flatness can be described in terms of the notion of absolute equivalence defined by E. Cartan.

In this chapter we adopt a somewhat different geometric point of view, relying on a Lie-Bäcklund framework as the underlying mathematical structure. This point of view was originally described by Fliess et al. and is related to the work of Pomet et al. on "infinitesimal Brunovsky forms" (in the context of feedback linearization). It offers a compact framework in which to describe basic results and is also closely related to the basic techniques that are used to compute the functions that are required to characterize the solutions of flat systems (the so-called flat outputs).

Applications of flatness to problems of engineering interest have grown steadily in recent years. It is important to point out that many classes of systems commonly used in nonlinear control theory are flat. As already noted, all controllable linear systems can be shown to be flat. Indeed, any system that can be transformed into a linear system by changes of coordinates, static feedback transformations (change of coordinates plus nonlinear change of inputs), or dynamic feedback transformations is also flat. Nonlinear control systems in "pure feedback form", which have gained popularity due to the applicability of backstepping to such systems, are also flat. Thus, many of the systems for which strong nonlinear control techniques are available are in fact flat systems, leading one to question how the structure of flatness plays a role in control of such systems.

It is true that any flat system can be feedback linearized using dynamic feedback (up to some regularity conditions that are generically satisfied). However, flatness is a property of a system and does not imply that one intends to then transform the system, via a dynamic feedback and appropriate changes of coordinates, to a single linear system. Indeed, the power of flatness is precisely that it does not convert nonlinear systems into linear ones. When a system is flat it is an indication that the nonlinear structure of the system is well characterized and one can exploit that structure in designing control algorithms for motion planning, trajectory generation, and stabilization. Dynamic feedback linearization is one such technique, although it is often a poor choice if the dynamics of the system are substantially different in different operating regimes.

Another advantage of studying flatness over dynamic feedback linearization is that flatness is a *geometric* property of a system, independent of coordinate choice. Typically when one speaks of linear systems in a state space context, this does not make sense geometrically since the system is linear only in certain choices of coordinate representations. In particular, it is difficult to discuss the notion of a linear state space system on a manifold since the very definition of linearity requires an underlying linear space. In this way, flatness can be considered the proper geometric notion of linearity, even though the system may be quite nonlinear in almost any natural representation.

This chapter provides a self-contained description of flat systems. It introduces the fundamental concepts of equivalence and flatness in a simple geometric framework. This is essentially an open-loop point of view.

2. Equivalence and Flatness

2.1. Control Systems as Infinite Dimensional Vector Fields

A system of differential equations

$$\dot{x} = f(x), \quad x \in X \subset \mathbb{R}^n \qquad (1)$$

is by definition a pair (X, f), where X is an open set of $\mathbb{R}^n$ and f is a smooth vector field on X. A solution, or *trajectory*, of (1) is a mapping $t \mapsto x(t)$ such that

$$\dot{x}(t) = f(x(t)) \quad \forall t \geq 0.$$

Notice that if $x \mapsto h(x)$ is a smooth function on X and $t \mapsto x(t)$ is a trajectory of (1), then

$$\frac{d}{dt} h(x(t)) = \frac{\partial h}{\partial x}(x(t)) \cdot \dot{x}(t) = \frac{\partial h}{\partial x}(x(t)) \cdot f(x(t)) \quad \forall t \geq 0.$$

For that reason the *total derivative*, i.e., the mapping

$$x \mapsto \frac{\partial h}{\partial x}(x) \cdot f(x)$$

is somewhat abusively called the "time-derivative" of h and denoted by $\dot{h}$.

We would like to have a similar description, i.e., a "space" and a vector field on this space, for a control system

$$\dot{x} = f(x,u), \qquad (2)$$

where f is smooth on an open subset $X \times U \subset \mathbb{R}^n \times \mathbb{R}^m$. Here f is no longer a vector field on X, but rather an *infinite collection* of vector fields on X parameterized by u: for all $u \in U$, the mapping

$$x \mapsto f_u(x) = f(x,u)$$

is a vector field on X. Such a description is not well-adapted when considering dynamic feedback.

It is nevertheless possible to associate to (2) a vector field with the "same" solutions using the following remarks: given a smooth solution of (2), i.e., a mapping $t \mapsto (x(t), u(t))$ with values in $X \times U$ such that

$$\dot{x}(t) = f(x(t), u(t)) \quad \forall t \geq 0,$$

we can consider the *infinite* mapping

$$t \mapsto \xi(t) = (x(t), u(t), \dot{u}(t), \ldots)$$

taking values in $X \times U \times \mathbb{R}_m^\infty$, where $\mathbb{R}_m^\infty = \mathbb{R}^m \times \mathbb{R}^m \times \ldots$ denotes the product of an infinite (countable) number of copies of $\mathbb{R}^m$. A typical point of $\mathbb{R}_m^\infty$ is thus of the form

$(u^1, u^2, \ldots)$ with $u^i \in \mathbb{R}^m$. This mapping satisfies

$$\dot{\xi}(t) = (f(x(t), u(t)), \dot{u}(t), \ddot{u}(t), \ldots) \quad \forall t \geq 0,$$

hence it can be thought of as a trajectory of the *infinite* vector field

$$(x, u, u^1, \ldots) \mapsto F(x, u, u^1, \ldots) = (f(x, u), u^1, u^2, \ldots)$$

on $X \times U \times \mathbb{R}_m^\infty$. Conversely, any mapping

$$t \mapsto \xi(t) = (x(t), u(t), u^1(t), \ldots)$$

that is a trajectory of this infinite vector field necessarily takes the form $(x(t), u(t), \dot{u}(t), \ldots)$ with $\dot{x}(t) = f(x(t), u(t))$, hence corresponds to a solution of (2). Thus F is truly a vector field and no longer a parameterized family of vector fields.

Using this construction, the control system (2) can be seen as the data of the "space" $X \times U \times \mathbb{R}_m^\infty$ together with the "smooth" vector field F on this space.

Notice that, as in the uncontrolled case, we can define the "time-derivative" of a smooth function $(x, u, u^1, \ldots) \mapsto h(x, u, u^1, \ldots, u^k)$ depending on a *finite* number of variables by

$$\dot{h}(x, u, u^1, \ldots, u^{k+1}) := Dh \cdot F$$

$$= \frac{\partial h}{\partial x} \cdot f(x, u) + \frac{\partial h}{\partial u} \cdot u^1 + \frac{\partial h}{\partial u^1} \cdot u^2 + \cdots.$$

The above sum is *finite* because h depends on finitely many variables.

See the book of Zharinov for a rigorous statement of the underlying topology and differentiable structure of $\mathbb{R}_m^\infty$ to be able to speak of smooth objects.

We are now in position to give a formal definition of a system:

Definition 1: A system *is a pair* $(\mathfrak{M}, F)$ *where* $\mathfrak{M}$ *is a smooth manifold, possibly of infinite dimension, and* F *is a smooth vector field on* $\mathfrak{M}$.

Locally, a control system looks like an open subset of $\mathbb{R}^\alpha$ (α not necessarily finite) with coordinates $(\xi_1, \ldots, \xi_\alpha)$ together with the vector field

$$\xi \mapsto F(\xi) = (F_1(\xi), \ldots, F_\alpha(\xi))$$

where all the components F_i depend only on a finite number of coordinates. A *trajectory* of the system is a mapping $t \mapsto \xi(t)$ such that $\dot{\xi}(t) = F(\xi(t))$.

We saw in the beginning of this section how a "traditional" control system fits into our definition. There is nevertheless an important difference: we lose the notion of *state dimension*. Indeed

$$\dot{x} = f(x,u), \quad (x,u) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m \qquad (3)$$

and

$$\dot{x} = f(x,u), \quad \dot{u} = v \qquad (4)$$

now have the same description $(X \times U \times \mathbb{R}_m^\infty, F)$, with

$$F(x,u,u^1,...) = (f(x,u),u^1,u^2,...),$$

in our formalism: $t \mapsto (x(t),u(t))$ is a trajectory of (3) if and only if $t \mapsto (x(t),u(t),\dot{u}(t))$ is a trajectory of (4). This situation is not surprising since the state dimension is of course not preserved by dynamic feedback. On the other hand we will see there is still a notion of *input dimension*.

Example 1: (The trivial system). *The* trivial system $(\mathbb{R}_m^\infty, F_m)$, *with coordinates* $(y, y^1, y^2,...)$ *and vector field*

$$F_m(y, y^1, y^2,...) = (y^1, y^2, y^3,...)$$

describes any "traditional" system made of m chains of integrators of arbitrary lengths, and in particular the direct transfer $y = u$.

In practice we often identify the "system" $F(x,\bar{u}) := (f(x,u),u^1,u^2,...)$ with the "dynamics" $\dot{x} = f(x,u)$ which defines it. Our main motivation for introducing a new formalism is that it will turn out to be a natural framework for the notions of equivalence and flatness we want to define.

Remark 1: *It is easy to see that the manifold $\mathfrak{M}$ is finite-dimensional only when there is no input, i.e., to describe a* determined *system of differential equations one needs as many equations as variable.*

In the presence of inputs, the system becomes underdetermined, *there are more variables than equations, which accounts for the infinite dimension.*

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Biographical Sketches

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