ANALYSIS OF CHAOTIC SYSTEMS

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Summary

The notion of chaos was introduced into science in the 1970s. Since then a lot of evidences of chaotic behavior in natural and artificial systems have been observed. Methods of chaotic systems analysis have turned into an efficient toolbox for understanding instabilities of various origins. In this chapter the most important notions and results of chaos theory are presented, such as definitions and examples of chaotic systems; criteria of chaos, providing basis for qualitative judgment about chaotic behavior; quantitative indices measuring chaos related properties of dynamical systems.

1. Introduction

In the second half of the 20th century the scientific vocabulary was enriched with a new term: “chaos”. Chaotic system is a deterministic dynamical system exhibiting irregular, seemingly random behavior. Two trajectories of a chaotic system starting close to each other will diverge after some time (so-called “sensitive dependence on initial conditions”). Mathematically chaotic systems are characterized by local instability and global boundedness of the trajectories. Since local instability of a linear system implies unboundedness (infinite growth) of its solutions, chaotic system should be necessarily nonlinear, i.e. should be described by a nonlinear mathematical model.

Chaotic systems provide researchers with a new way to describe uncertainty, without appealing to probabilistic concepts. Unlike conventional paradigm of deterministic models, where model is believed to be suitable for prediction with an arbitrarily long horizon given the current system state, prediction error of a chaotic system is growing with time exponentially because of its instability. Therefore, chaotic model is good for prediction only for a restricted time, the prediction horizon depending on the required prediction accuracy.

Since 1975, when the term ``chaos'' was introduced by T. Li and J. Yorke into scientific discussions and publications, chaotic phenomena and chaotic behavior have been observed in numerous natural and model systems in physics, chemistry, biology, ecology, etc. Paradigm of chaos allows us to better understand inherent properties of natural systems. Methods of chaotic systems analysis are mature enough today and allow for broad usage of nonlinear models in science and technologies. An advantage of nonlinear models is their ability to describe complex behavior with a small number of variables and parameters. Among perspective areas of engineering applications are lasers and plasma technologies, mechanical and chemical engineering, system engineering and telecommunications.

A key property of chaotic systems is its instability: sensitive dependence on initial conditions. An important consequence is high sensitivity with respect to changes of external disturbance (input or controlling action). It means that small portions of control may produce large variations in systems behavior. Possibilities of controlling complex behavior by means of small control open new horizons both in science and in technology.

In addition to instability analysis, based on studying individual trajectories, statistical approach can be used based on studying ergodic properties of ensembles of trajectories. Such an approach allows describe the integral, typical properties of trajectories, eliminating exceptional, atypical trajectories.

(see *Control of chaotic systems*, *Control of bifurcations and chaos*)

2. Notions of Chaos

2.1. From Oscillations to Chaos: Evolution of the Concept of Oscillations

Chaotic systems are understood as those exhibiting complex, irregular oscillatory behavior. Unlike regular oscillations composed of one or several periodic components and having fixed amplitudes and finite frequency spectra, chaotic oscillation are characterized by changing, “floating” of both amplitudes and frequencies of oscillations.

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Since oscillatory phenomena appear in many natural and technical systems, models and methods of their description are in process of sustained development. Before the beginning of 20th century models of oscillatory systems were usually chosen as linear differential equations, e.g.

$$\ddot{y}(t)+\omega^2 y(t)=0, \quad \omega\in\Re, \quad 0\le t<\infty. \tag{1}$$

Solutions of (1) are simple harmonic oscillations:

$$y(t)=A_0\sin\omega t+A_1\cos\omega t \tag{2}$$

where ω is circular frequency, $T=2\pi/\omega$, is period, $A=\sqrt{A_0^2+A_1^2}$ is amplitude. Amplitude depends on initial conditions: $A_1=y(0)$, $A_0=\dot{y}(0)/\omega$ (see Fig. 1, *a*, for $\omega=1$). The solution of (2) continuously depends on initial conditions: a small change of $y(0),\dot{y}(0)$ leads to a small change of $y(t)$ over the whole time semi-axis $0\le t<\infty$. Besides, the frequency spectrum of the function (2) consists of one point $\omega/2\pi$ (see Fig. 1, *b*), i.e. it is discrete.

To describe more complex oscillations a composition of models (1) having different frequencies $\omega_1,\dots,\omega_r$ can be employed. For example, series connection of two models (1) is described by equations

$$\ddot{y}_1(t)+\omega_1^2 y_1(t)=0$$
$$\ddot{y}_2(t)+\omega_2^2 y_2(t)=y_1(t).$$

This system has particular solutions of the form $y_2(t)=A_1\sin\omega_1 t+A_2\sin\omega_2 t$, where coefficients A_1, A_2 depend on initial conditions. If the frequencies $\omega_1,\dots,\omega_r$ are commensurable (are integer multiples of some frequency ω_0), then the solutions will be periodic with period $2\pi/\omega_0$ (see Fig. 2 for $r=3$, $\omega_1=1$, $\omega_2=2$, $\omega_3=4$). Otherwise, if ω_i are incommensurable, the oscillations are *quasiperiodic* rather than periodic (see Fig. 3, where $r=2$, $\omega_1=1$, $\omega_2=5/\pi$). In both cases the solution depends continuously on initial conditions and its spectrum is a finite discrete set.

Note that it is almost impossible to distinguish between periodic and quasiperiodic functions (i.e. between rational and irrational frequency ratio) either by eye or by a measurement device since measurements have a finite accuracy.

Physics and technology demands at the turn of the 19th and 20th centuries were not satisfied by linear models of oscillations. Fundamentals of new mathematical apparatus – *nonlinear oscillations theory* were mainly developed by H. Poincaré, B. Van der Pol, A.A. Andronov, E. Hopf, N.M. Krylov and N.N. Bogolubov. The core concept of

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nonlinear oscillations theory is *limit cycle*, that is, a periodic trajectory attracting other trajectories starting close to it. Typical examples of nonlinear differential models with limit cycles are the *Van der Pol equation*

$$\ddot{y} + \varepsilon\left(y^2 - 1\right)\dot{y} + \omega^2 y = 0, \qquad (3)$$

where $\varepsilon > 0$; *Duffing equation*

$$\ddot{y} + p\dot{y} - qy + q_0 y^3 = 0, \qquad (4)$$

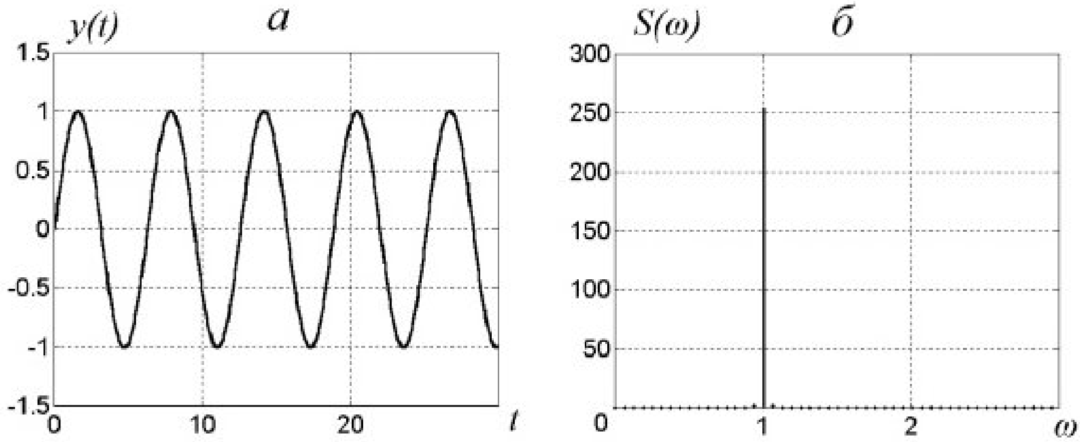

Figure 1: Simple harmonic oscillations ($\omega = 1$).

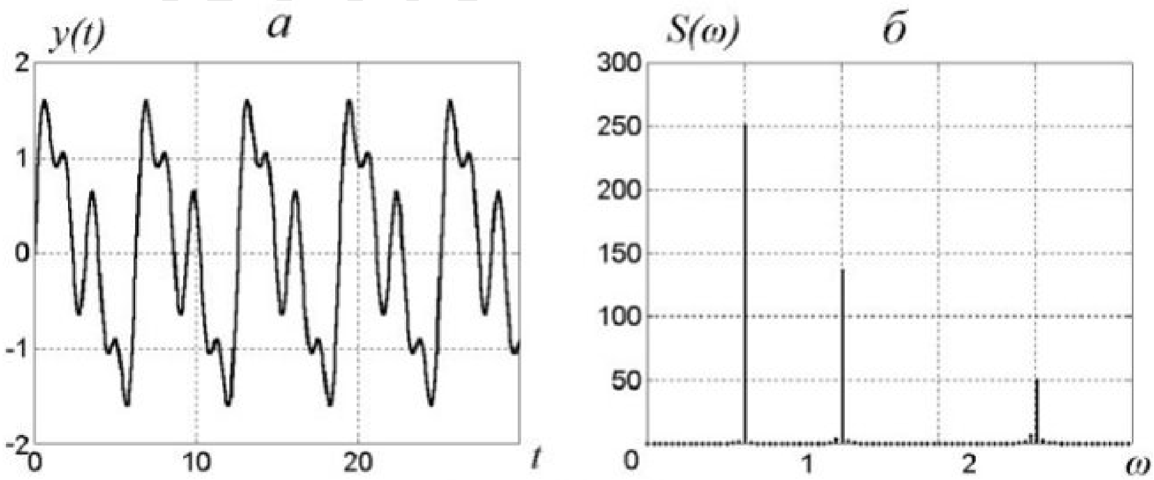

Figure 2: Periodic oscillations ($\omega_i = 1, 2, 4$).

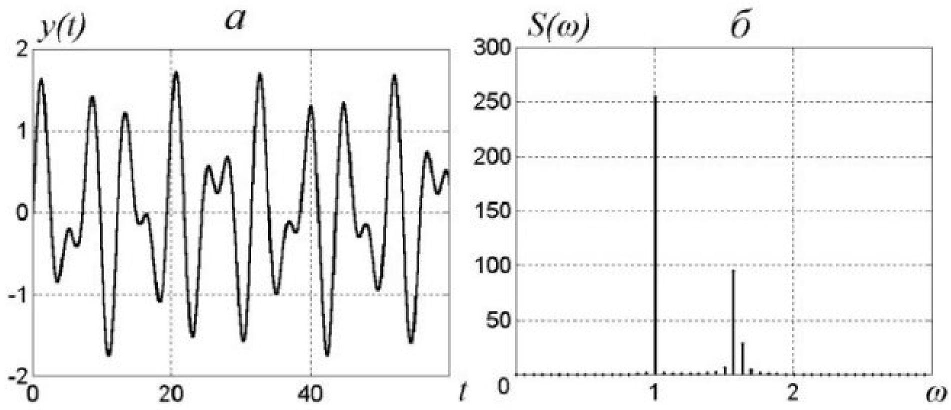

Figure 3: Quasiperiodic oscillations ($\omega_i = 1,5/\pi$).

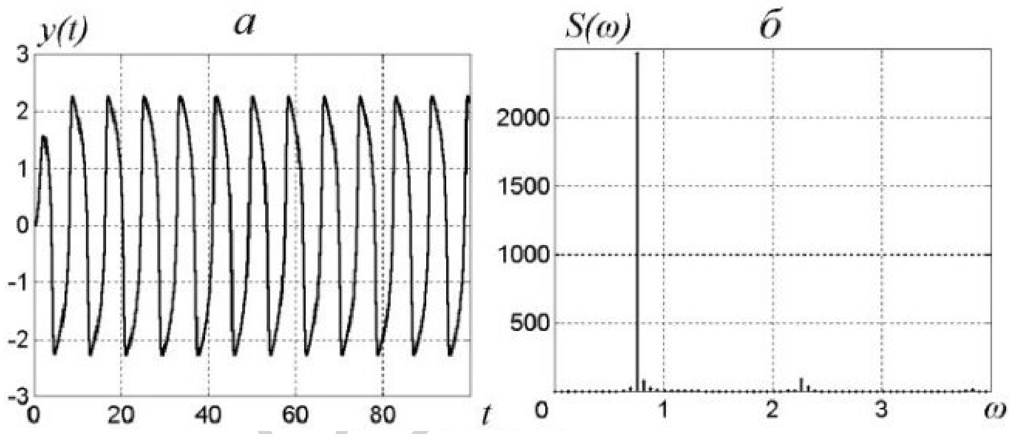

Figure 4: Limit cycle and its spectrum.

where $p > 0$, $q > 0$, $q_0 > 0$; 2nd-order relay system

$$\ddot{y} + p\dot{y} + qy - sign(y) = 0. \qquad (5)$$

Even simple nonlinear models allow us to describe oscillations of complex shape, e.g. relaxation oscillations (close to square waves). Spectrum of a limit cycle consists of an infinite number of frequencies, multiples of some minimum frequency (see Fig. 4), where the solution of system (3) for $\varepsilon = 2.5$, $\omega = 1$, $y(0) = 0.5$, $\dot{y} = 0$ and its spectrum are shown.

During a few decades it was believed that linear models and nonlinear models possessing limit cycles describe all possible types of oscillatory behavior. However, in the 1940s-1950s mathematicians M. Cartwright and J. Littlewood discovered the existence of bounded aperiodic forced solutions in some systems of second order while

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S. Smale introduced the “horseshow” example and demonstrated that complicated aperiodic behavior is not exceptional. A real revolution was inspired by the paper of the E. Lorenz published in 1963, who numerically studied a simple nonlinear model of 3rd order:

$$\begin{cases} \dot{x} = \sigma\left(y - x\right), \\ \dot{y} = rx - y - xz, \\ \dot{z} = -bz + xy, \end{cases} \tag{6}$$

now called *Lorenz system*. System (6) appeared as a simplified model of atmospheric turbulence (described by Navier-Stokes equations). Solutions of (6) for some parameter values look like irregular oscillations (see, e.g. (Fig. 5 for the case $\sigma = 10$, $r = 97$, $b = 8/3$). Trajectories in the state space may approach a limit set (so-called *attractor* having quite an intricate shape. Such systems received significant interest of physicists, mathematicians and engineers after publishing in 1971 the paper of D. Ruelle and F. Takens, who termed such limit sets “strange attractors” and publishing in 1975 the paper by T. Li and J. Yorke, who introduced the term “chaos” for the whole phenomenon.

Note that one of the central results of the paper by T. Li and J. Yorke is a special case of the theorem due to Ukrainian mathematician A.N. Sharkovsky published in 1964. Important contributions into the methods of chaotic systems analysis were made in the 1960s by Russian (Soviet) researchers A.N. Kolmogorov, V.I. Arnold, Ya.G. Sinai, D.A. Anosov, V.K. Melnikov, Yu.I. Neimark, L.P. Shilnikov, among other contributions in the field.

Later chaotic behavior was found in numerous model and real world systems in mechanics, laser physics, biophysics, chemistry, biology and medicine, electronic circuits, etc. It was established by newly developed methods of chaotic systems analysis that chaos is not an exceptional form of nonlinear behavior. Loosely speaking, chaotic motion arises as soon as the system trajectories are locally unstable yet globally bounded. Small initial displacement of chaotic trajectories does not remain small infinitely long but grows exponentially as time evolves. Frequency spectrum of a chaotic system is continuous (see Fig. 5, *b*). In many cases such aperiodic, irregular oscillations better correspond to the properties of processes in real systems. On the other hand, it is impossible to distinguish between chaotic and quasiperiodic functions by eye or by a measurement device, because of finite accuracy of the physical measurements.

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Bibliography

A variety of websites containing chaos related links and references can be found on the WEB. A well known bibliography, containing more than 12000 references in 2001 was collected by P. Beckmann at the University of Mainz at http://nld.physik.uni-mainz.de/bib_main.htm

Books

Glass L. and M.Mackey (1988). *From Clocks to Chaos: The Rhythms of Life*. Princeton University Press. [A clear and readable introduction into nonlinear oscillations and waves including chaotic oscillations and waves as applied to biological problems.]

Guckenheimer J. and P.Holmes (1983). *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. Springer-Verlag. [Classical textbook of nonlinear dynamics and mathematical chaos theory.]

Kapitaniak T. (2000). *Chaos for Engineers*. 2nd edition. Springer-Verlag. [Simple introductory exposition of some methods from chaotic dynamics including control and synchronization of chaos.]

Mandelbrot B.(1982). *The Fractal Geometry of Nature*. W.H.Freeman, San Francisco. [The classical treatise on fractal sets and their applications.]

Moon F. (1992). *Chaotic and Fractal Dynamics. An Introduction for Applied Scientists and Engineers*. Wiley. [A readable and detailed exposition of the chaos theory with many methods and examples, including first ideas of control of chaos.]

Nayfeh A. and B. Balakrishnan (1995) *Applied Nonlinear Dynamics*. Wiley. [A comprehensive collection of concepts, methods and numerical procedures from the nonlinear dynamics field. Brief account of chaos control methods.]

Parker, T.S. and L.O. Chua, (1989), *Practical Algorithms for Chaotic Systems*, (Springer-Verlag, New York). [Basics of chaos theory and readable exposition of numerical methods for analysis of chaotic systems.]

Strogatz S. (1994). *Nonlinear Dynamics and Chaos*. Reading, Addison-Wesley. [A concise and deep introduction into the field of nonlinear dynamics.]

Wiggins S. (1990). *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (Springer-Verlag, New York).

Journal papers

Lorenz, E.N., (1963), "Deterministic nonperiodic flow," *Journal of the Atmospheric Sciences*, vol. **20**, pp. 130–141. [Classical paper studying properties of "butterfly attractor" and introducing methods of analyzing chaotic behavior.]

Ruelle D. and F.Takens, (1971), "On the nature of turbulence," *Comm. Math. Physics*, vol. **20**(2), pp.167–192. [A seminal paper introducing the term "strange attractor".]

Li T. and J.A.Yorke, (1975), "Period three implies chaos," *Amer. Math. Monthly*, vol. **82**, pp.985–992. [A seminal paper introducing the term "chaos".]

Sharkovsky, A. N. Coexistence of cycles of a contiuous map of the line into itself Int. J. of Bifurcation and Chaos, vol. 5, N 5, pp.1263–1273, 1995. [Precursor of the paper by Li-Yorke, containing a more general periodicity result.]

Grassberger, P.; Procaccia, I. Measuring the strangeness of strange attractors, Physica D., vol. 9, pp.189–208, 1983. [A paper introducing numerical algorithms for evaluating fractal dimensions and entropies.]

Melnikov V.K. (1963) On stability of center for time periodic bifurcations. Trans. Moscow Math. Soc. vol.12, pp.1–57. [A classical paper containing foundations of *Melnikov method*.]

Shilnikov L.P. (1965) The existence of a denumerable set of periodic motions. Sov. Math. Dokl. vol.6, pp.163–166. [A classical paper where a theorem of chaos with homoclinic orbits is established.]

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Biographical Sketch

Alexander Lvovich Fradkov born May 22, 1948; received the Diploma degree in mathematics from the Faculty of Mathematics and Mechanics of St. Petersburg State University (Dept. of Theoretical Cybernetics) in 1971; Candidate of Sciences (Ph.D.) degree in Engineering Cybernetics from St. Petersburg Mechanical Institute (now - Baltic State Technical University - BSTU) in 1975 and Doctor of Sciences degree in Control Engineering in 1986 from St. Petersburg Electrotechnical Institute.

From 1971 to 1987 he occupied different research positions and in 1987 became Professor of Computer Science with BSTU. Since 1990 he has been the Head of the Laboratory for Control of Complex Systems of the Institute for the Problems of Mechanical Engineering of Russian Academy of Sciences. He is also a part time professor with the Faculty of Mathematics and Mechanics of St. Petersburg State University (Dept. of Theoretical Cybernetics). His research interests are in fields of nonlinear and adaptive control, control of oscillatory and chaotic systems and mathematical modeling. He is also working in the borderland field between Physics and Control (Cybernetical Physics). Dr. Fradkov is coauthor of more than 300 journal and conference papers, 9 patents, 14 books and textbooks, including the recent ones as follows: INTRODUCTION TO CONTROL OF OSCILLATIONS AND CHAOS (with A.Yu. Pogromsky, Singapore: World Scientific, 1998), NONLINEAR AND ADAPTIVE CONTROL OF COMPLEX SYSTEMS (with I.V. Miroshnik and V.O.Nikiforov, Dordrecht: Kluwer, 1999), SELECTED CHAPTERS OF AUTOMATIC CONTROL THEORY with MATLAB examples (with B.R. Andrievsky, St. Petersburg: Nauka, 1999, in Russian), ELEMENTS OF MATHEMATICAL MODELING IN SOFTWARE ENVIRONMENTS MATLAB 5 AND SCILAB (with B.R. Andrievsky, St. Petersburg: Nauka, 2001, in Russian). Dr. Fradkov is the Vice-President of the St.Petersburg Informatics and Control Society since 1991, Member of the Russian National Committee of Automatic Control, IEEE Senior Member. Dr. Fradkov was Co-Chairman of 1st-9th International Baltic Student Olympiades on Automatic Control in 1991-2002; NOC Chairman of the 1st and 2nd International IEEE-IUTAM Conference "Control of Oscillations and Chaos" in 1997 and 2000; NOC Chairman of the 5th IFAC Symposium on Nonlinear Control Systems (NOLCOS'01). He was an associate editor of European Journal of Control (1998-2001); a member of IEEE Control Systems Society Conference Editorial Board (1998-2002); a member of the IFAC Technical Committees on Education and Nonlinear Control. Dr. Fradkov was awarded William Girling Watson Scholarship in Electrical Engineering (University of Sydney, 1995) and JSPS Fellowship for Research in Japan in 1998-1999. During 1991-2001 he visited and gave invited lectures in more than 60 Universities of 20 countries.

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